

# THE MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

63 COLERAINK ROAD, BLACKHEATH, LONDON, S.E. 3.

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# THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession: from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves, by way of amends, to be a help and an ornament therunto."—BACON (Preface, *Maxims of Law*).

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## THE EXECUTIVE COMMITTEE.

IN view of the difficulty of arranging for meetings under existing conditions, the Council has set up an Executive Committee to conduct the business usually transacted by the Council. This Committee consists of the President, the Treasurer, the Secretaries, the Librarian, the Editor of the *Mathematical Gazette*, the Chairman and the Secretary of the Teaching Committee, together with Mr. Siddons, Mr. Hope-Jones and Miss Punnett.

This Committee met in October to discuss the problem of conducting the Association in war-time. The following is a brief resumé of the main decisions made :

(i) It was decided that the Annual Meeting should not be held in January 1940.

(ii) It was proposed to maintain, as far as possible, the issue of five numbers of the *Mathematical Gazette* each year, though it was realised that some reduction in size would be inevitable.

(iii) It was hoped that as many members as possible would maintain their subscriptions : in the case of members on service the Association would be prepared, on request, to store copies of the *Mathematical Gazette* and to hand them over later.

(iv) In the case of Junior Members called up for military service before the end of the first year of membership, it was decided that the second year of reduced subscription could be held over till the end of such service, if the member desired to do so.

(v) It was decided that, notwithstanding any provisions in the Rules of the Association, the President and other officers elected in January 1939 should be asked to continue in office until further notice.

# MATHEMATICS AND THE ENGINEERING STUDENT: SOME GENERAL CONSIDERATIONS.\*

BY W. G. BICKLEY.

My remarks this morning will be devoted mainly to general considerations and to principles. Dr. Topping and Mr. Verity are to deal with the courses in detail, but behind all that they say there must be a background of principle, and I hope that a statement of my creed will not find them active dissenters.

I take as my text a quotation from Ruskin which I placed on the flyleaf of a little book I wrote in 1925,† and of the truth of which I am as convinced as ever—"Know what you have to do, and do it; . . . for I believe that failure is less frequently attributable to either insufficiency of means or impatience of labour, than to a confused understanding of the thing actually to be done." It does not matter much if Dr. Topping, Mr. Verity, and you disagree with my subsequent remarks—although I naturally hope that the disagreement will at least not be too violent—for my immediate purpose will have been served if I have, in the spirit of my quotation, made you face the questions:

- (I) *The problem.* What are we trying to do?
- (II) *The short-range problem.* How far, and how best, can it be done under present conditions?
- (III) *The long-range problem.* What modifications in the conditions should be striven for in order that our aim can be better accomplished?

My own answers to these questions will be given by implication only.

## Technical Education.

First of all I want to consider the wider aspect, *i.e.* technical education as a whole, and I begin with a short historical sketch.

Education was for centuries the prerogative of the priests, and later of the aristocracy. In both periods, however, opportunities existed for the sons of poorer people to obtain the advantages of education, as is evident in the origins of many of our most famous seats of learning, but the richer elements of the community gradually diverted these to their own enjoyment, and, what is more important, always directed the orientation of the studies pursued in the schools and universities. Concurrently with all this, the crafts—the forerunners of present-day technology—had a system of training and education by apprenticeship, practised, for instance, by the

\* An address delivered to the Mathematical Section of the Board of Education Engineering Summer School at Oxford; the Board is, however, neither responsible for, nor necessarily in agreement with, the views expressed.

† It may also be regarded as the presentation of a point of view diametrically opposed to that of Mr. Dockeray in his article in the *Gazette* for October, 1938.

† *Engineering Applications of Mathematics* (Pitmans).

medieval Guilds. The industrial revolution disturbed the relative equilibrium of these two systems, by introducing new forces which have been causing a gradual transition towards a new position of relative equilibrium, but a century has not been long enough for them to overcome the inertia and resistance which they have encountered.

The manifestations of this trend became evident in this country a century or so ago, in the rise of Sunday and night schools for the operatives of the factories which were becoming ever more numerous. At first these schools were "cultural", and not "technical", but as the ability to read and write became more widely disseminated, and the complexity of machinery and industrial processes increased, first science, and then technology, began to permeate their curricula. It would take too long to review the work of the Science and Art Department, and the effect of Whitworth Scholarships, but it is to be noted that technical education struggled into existence as an offshoot of the movement which was in the first instance an attempt by the working classes to share in the educational advantages which they saw their betters enjoying, and *not* as a development of their own apprenticeship or Guild system. Technical education grew as an appendage of the "school" system of education, and was in some quarters—and to-day still is—regarded as a cancer or as a foreign body in that system.

Since the last war things have changed, and in a piecemeal, haphazard, and uncoordinated way technical schools and colleges have multiplied, but there are few signs that either the authorities or the public have yet attained a sane and well-thought-out attitude to technical education. Even the advocacy of "Technical High Schools" with the "same status" as "Grammar Schools" in the Spens Report is mixed up with so much that does not seem to receive general assent that one cannot be sanguine that a new era has been inaugurated. The report will not kill at once the idea that lawyers and stockbrokers are so much more respectable than engineers and other craftsmen!

Internal as well as external resistances have had to be overcome. The great distinction between the book learning of the schools and the apprentice learning of the Guilds remained, and we all know how the older men in engineering and in business remained unconvinced that anything of relevance to the job in hand could be learned anywhere but in the workshop or in the office. It is not so long ago that a degree in engineering would be regarded rather as a *disqualification* by employers.

In view of this picture of its origin and growth, it is not to be expected that technical education has so far been an unqualified success. If internal causes of shortcoming are examined, it will be seen that they are mainly two:

- (1) it has not been sufficiently "technical",
- (2) it has not been sufficiently "education".

These two points will be exemplified when I come to consider mathematics as an element of technical education, but I may as well say here that I hold very strongly the opinion that the latter need not—and should not—be any the less education for being technical!

It is not possible, of course, entirely to dissociate a discussion of technical education from the education given in the primary and secondary schools. So long as the attitude of the secondary schools is predominantly one which sees little if any educational value in technical studies, the technical schools are faced with the dual problem of having to produce a change of outlook in the students as well as to increase their stock of knowledge. The Spens Report is right in emphasising the question of *status*, and public opinion must be educated upon this question. Our attitude towards our work can greatly help here.

#### *Mathematics in Technical Education.*

The “rapid mathematisation of the sciences”, to which Professor Hogben referred so forcibly in his presidential address to the London Branch of the Mathematical Association,\* means that few branches of technical education can dispense entirely with mathematics. Naturally the demands made upon mathematics by the different branches of technology vary widely in amount and scope, but certain general principles should be of universal application. Both by reason of their total numbers, and the range of mathematics which they can usefully learn to apply, engineering students form, as regards our subject, the most important group, and it is natural that their demands and needs should have received more attention than those of other groups. Mainly for them was the trail first blazed, and technical students generally have benefited, and will continue to benefit, by advances made in the first instance for engineers. But I would like still to use the wider term, “technical”, and to define “technical students” from our point of view as those students who study mathematics primarily in order to be able to apply it to some technical subject or subjects, *i.e.* who have some extra-mathematical aim. It is just sheer nonsense to shut our eyes to this ulterior aim, or to complain that these students do not come to us prepared to study mathematics for its own sake. The pure mathematician behaving as such an egotistical ostrich is still to be found in our universities and secondary schools, and at least in the former has not given the engineering student a fair deal.

The first principle I wish, then, to urge is that any course in mathematics for technical students must be planned with their particular needs and interests constantly in mind. To do this the teacher must make himself familiar with the types of application which his students will meet in their other classes. Lack of such familiarity will make it difficult for the teacher to retain the interest of his students, or to use most effectively the time at his disposal. It will prevent him from taking full advantage of the knowledge

\* *Math. Gazette*, XXII, 105 (May, 1938).

which his students are acquiring in these other classes, and from exploiting this knowledge for his own mathematical ends. In planning his course he should convince himself that there is a good reason for the inclusion of any mathematical topic. The older textbooks written for schools contain a deal of lumber, and the writers of the flood of books on mathematics for technical students which has drenched the market have not as a rule been sufficiently radical or discriminating to discard all this lumber. Broadly speaking, no topic should be included unless it is

- (a) necessary in some technical application, or
- (b) a necessary step in building up or coordinating knowledge which will later be justified by (a).

The course must be definitely, frankly, and intentionally technical!

Secondly, our teacher will consistently be on his guard lest his lessons degenerate into the mere giving of useful results—which is unfortunately about all that our engineering colleagues seem often to demand. If the course is also to be educative, it must become clear to the students that the mathematical topics are closely inter-related and fit together into a coordinated structure of knowledge. From the educative point of view it is also imperative, however limited the time at our disposal may be, that the universality of mathematics should be demonstrated by applications *outside* the immediate technical scope of the particular group of students. It is eminently desirable that the students should come to realise that mathematics is not only one of the tools which man has invented and laboriously developed in order to accomplish what he desired to accomplish, but is also part of our common cultural heritage. Nor is it too much to hope that some realisation, however dim, of the aesthetic appeal of pure mathematics for its own sake may be attained.

#### *The technical teacher.*

All this, I expect some of you are thinking, doesn't sound particularly easy. It isn't! No job really worth doing is! But it is a job so *very* well worth doing—and any difficulty additional to that of the secondary school teacher or the university don should endow it with additional interest. It is also, in our present civilisation, a vital piece of social service. The technical teacher should be—and I maintain that the really effective teacher of technical students is—the cream of our profession. Except in so far as it may constitute a tacit admission of the fact that the first-class teacher is born and not made, the present lack of encouragement or special incentive to become such teachers, and the attitude of the mathematics departments of most of our universities and training colleges towards technical education, would seem to imply that my view is not generally held. That does not lessen my conviction of its truth, and I should like to congratulate the originators and organisers of this Summer School, which, as far as I am aware, is the one and only effort being

made to assist and inspire the technical teachers in their work, and to say how much I appreciate the honour of being allowed to assist, even in a small way, in this enterprise. But much—very much—more is needed. Especially is it necessary that the universities and training colleges should recognise and cater for the needs of teachers who intend to enter technical schools. This is, of course, part of the wider problem of the place of technical education in the general educational system, but the teacher is at once the foundation and the power supply of any educational system, and without providing and maintaining a supply of suitable teachers no system can stand or progress.

*Some detailed applications of the principles.*

I will now attempt to crystallise some of the foregoing ideas by a few representative examples of how they can be carried into practice.

Taking first the junior course, it is pretty clear that an introduction to algebra and geometry must form its basis. Arithmetic also, since we cannot entirely rely upon that done in the elementary school. But, however necessary the revision of arithmetic, it should be postponed, so that the course may begin with something novel, in order to create and sustain interest. We must foster the illusion that our students are already engineers. Now the piece of mathematics which the engineer must most often do is to evaluate some quantity by use of a formula. This points the way to the introduction to algebra, which incidentally enables the revision of arithmetic to take place almost unnoticed.

But the points I most wish to urge here are two. First that *real* formulae should be used. The conviction of the *reality* of mathematics, that it does concern itself with *real* things, is most important. This is a cardinal point, and many teachers in secondary schools cannot leave the academic atmosphere of the university departments of pure mathematics sufficiently often to "put mathematics across" to their pupils. But with our students, we *must* create the illusion of reality. An engineer's pocket-book will provide all the material we need, for it is crammed with formulae of all orders of complexity, and their manipulation can take our students gradually through the whole gamut of elementary algebra.

Secondly, it is not enough merely to use real formulae, but they must be known and *felt* to be real. It is not enough to say:  $T = wl^2/8d$ , find  $T$  when  $w=2$ ,  $l=60$ ,  $d=7.5$ . Say rather, "We are going to design a suspension bridge. The span is to be 60 ft., the road will weigh 2 tons per foot run. The allowable sag is 7.5 ft. To make the chains strong enough we must know what the pull in them will be. You don't know how to do it? Well, you will one day, and here's the rule you will have to use." Explain carefully the meaning of the symbols, and discuss the units in which they would probably be measured. Then:

$$T = wl^2/8d.$$

"Does it look right? What happens to the tension if the load is increased? Does the formula tell us this? What would happen if we reduced the sag?" . . . and so on. A formula is firstly a statement about real things, a statement to be read, and understood, much as a musician can read a score and hear the music without playing a note. The subsequent arithmetic can be enlivened if the dimensions of one or two actual bridges (Hell Gate, for instance) are given.\* One cannot—and need not—embellish every exercise in this way, but it is well to make a rule, and to stick to it, that at this stage the meaning of the symbols, and the use, of every formula used in exercises be clearly stated. It is all very well for Bertrand Russell to say that "Pure mathematics is the subject in which you never know what you are talking about, nor whether what you are saying about it is true". That is not the engineers' outlook. He is not a pure mathematician! Starting with their outlook, you may get them ultimately to see the truths of the paradox, but not unless in their early steps our students *do* know what they are talking about, and have good reason for believing their statements about it to be true.

A little later, when dealing with negative quantities, questions on equilibrium using moments of forces form a particularly appropriate illustration of the inevitability of the rule of signs in multiplication.

Again, one will deal with graphs. Naturally, one starts with graphs of statistics, in order that the technique of plotting may be mastered without extraneous difficulties. But choose *real* statistics, and (this applies to all work on graphs) don't let the exercise stop when the graph is drawn. Let some use be made of every graph, some number read, or some deduction drawn. An (intentionally non-technical) instance is population statistics of England, Scotland, and Ireland in the nineteenth century, which immediately asks the question, "What happened in Ireland about 1850?" Collect a set of exercises, all with some similar point.

In leading up to the graph of a function, take early examples in which the values of the numbers to be plotted are obtained from a concrete problem by simple arithmetic. As an instance: the maximum rectangular pen that can be made with 100 yd. of wire netting and an existing straight fence.

Width	-	-	10	20	30 yd.
Length	-	-	80	60	40 yd.
Area	-	-	800	1200	1200 sq. yd.

After the plotting and discussion (what are the best dimensions? is there anything to be gained by considering the nonsensical case of zero width?) obtain the formula  $A = w(100 - 2w)$ , and write it on the graph.

Again,  $y = x^2$  is sure to be plotted. Show that it enables square roots to be read off, and let the graph be kept and *used* in questions on right-angled triangles, etc.

\* This example and some of the others were actually given in greater detail.

Another thing one should attempt to do is to let new mathematical topics arise as necessities in the solution of real problems. It may not always be easy to do this, but the type of thing I mean is illustrated in an article\* on the introduction of the exponential in the January 1926 number of the *Mathematical Gazette*. The starting point is the formulation of the belt-pulley friction problem, leading to the equation

$$dT/d\theta = \mu T,$$

connected with growth and decay problems in general, and compound interest in particular. This has been an eye-opener to some of my students at the Imperial College, who said they thought they had "done  $e$ " at school.

Some of you will say that all this takes up a lot of time. Granted. But our business is to exploit the time allotted to us to its maximum, and experience (my own and that of others) only strengthens my conviction that this is the way to achieve maximum results. The consciousness of purpose will make all the difference in the attitude to the subsequent routine exercises which are of course necessary for the mastery of technique, and will accelerate the rate at which technique is mastered.

I believe also that this realist attitude is the most effective for all students of mathematics—exceptions to this rule form a set of measure zero! One difficulty is often urged, that we are restricted to the "world of reality" of the students. Firstly, this difficulty obviously presses least on the technical teacher. Secondly, I do not believe that the difficulty itself is very real. I have always found students only too ready to be interested in other people's problems, provided they were told enough to appreciate and sufficiently to understand them. My own engineers cheerfully work exercises on the life of radium, on chemical reactions, on bacterial growth and epidemics, after being introduced to  $e$  via the belt and pulley, and gain added respect for  $e$  by so doing. A related difficulty is that our engineering colleagues will need the students to be familiar with the necessary mathematical technique *before* they meet the problem which my plan would use as an introduction. I unhesitatingly cut this Gordian knot. We must get in first! We must have the knowledge to present the engineering problem in outline before it is tackled in detail in the engineering classes. I have more than once been reproved for teaching "engineering", and not "sticking to my (mathematical) last". One good answer was provided by a student: "We learn in the engineering departments how things are done. We come to the maths. department to learn how they ought to be done." Nor is it waste of time for the student to meet the same thing twice—from *different* points of view. If only the engineers were able to teach mathematics *pari passu* with their engineering, as I maintain we must be to teach engineering *pari passu* with our mathematics, our task and theirs would be much easier.

\* *Math. Gazette*, XIII, 10.

And as far as they are *not* able to do this, surely the fault is with those who taught them mathematics.

One other point I would like you to consider. I have said that the traditional courses contain much lumber, vestigial remains of tradition which have not yet atrophied. In addition to ridding our courses of these, may it not also be desirable to consider the addition of more recent developments? Such, for instance, as nomograms and statistics in quite early stages, and others later.

I have not time to deal adequately with two other large questions. One is the place, amount, and type of geometry in the engineering course. My views on secondary school geometry are heretical to the point of blasphemy, but for engineering students I am sure that the approach to geometry should be in close contact with engineering drawing, and much of this in the early stages should be done in the mathematics classroom. The other point is the introduction to mechanics, too often left to our non-mathematical colleagues, with results which are sometimes unfortunate.

*Some conclusions.*

It will be inferred from my remarks that a major contribution to the solution of the short-range problem would be the dissemination of more technical knowledge among teachers of mathematics. A book I wrote in 1925 may perhaps be considered as a small contribution in this direction; I do not know that it has had any appreciable effect, for I suspect that the mathematicians fight shy of it because it is "engineering", while the engineers are afraid of it because it is "mathematics". Those of you who are members of the Mathematical Association—and I hope that is all of you—will know that an attempt on different lines, and much wider in scope, is being made at the instance of the sub-committee I mentioned, by an appeal to members to send in details of any application of mathematics which they happen to meet. Some interesting replies have been received, and a first batch will be published in the *October Gazette*.\* But the response has not so far been overwhelming and we hope that the appearance of the first batch will stimulate further replies. It is hoped that many of the applications will *not* be to engineering, for there are secondary school teachers who are eager to introduce the widest contact with reality into their classrooms.

The long-range problem, involving as it does not mathematics alone, not technical education alone, but our whole educational system, is too big to be profitably considered in our discussion this morning, but the discussion must be conducted with the consciousness of it in the background, and the conviction that to increase the knowledge of mathematics and the mathematician by engineers, and of engineering and the engineer by mathematicians, is one contribution we can make to its ultimate solution.

Finally, what should be the "educational" aims of a course in mathematics? The student should gain habits of accuracy, not

\* [This article was published in *Math. Gazette*, XXIII, pp. 338-9.]

merely in arithmetic, but in thought and expression; analysis of the data of a practical problem into assumptions, definitions, physical (or mechanical) laws; realisation that the simplification of the problem by idealisation is a legitimate step, and that complex problems must be simplified by division into simpler ones which can be tackled piecemeal; that mathematics is a language, one of the few universal languages, and the highest development so far in the attempt to express relations with clarity, precision, and economy—these are some of the aims.

In mathematics truth and falsehood are easier to disentangle than in other subjects. Although in many concerns of everyday life, and even in engineering, right and wrong are not so clearly certain, or error so definitely demonstrable, the mental attitude of the mathematician is still desirable. But what the psychologists term "carry-over" is not automatic. Unless we can exhibit the mathematical spirit in action in fields other than that of abstract mathematics, we shall not ensure the development of that mental attitude which is the aim of mathematics as an educational subject. W. G. B.

### GLEANINGS FAR AND NEAR.

**1291.** The influence of the great leader of the masses depends on a commonplace fact—it is the simplest thing in the world to lead a mob, easier than to solve a quadratic equation or compose a waltz.—Konrad Heiden, *One Man against Europe*, p. 29. [Per Mr. A. F. Mackenzie.]

**1292.** "As lines, so loves oblique may well  
Themselves in every angle greet :  
But ours so truly parallel,  
Though infinite can never meet."

—Andrew Marvell, "The Definition of Love." [Per Mr. C. E. Kemp.]

**1293.** RUSSIAN ARMS BUDGET FIGURE DOUBLED. Russia is to spend more than 40,000,000,000 roubles on arms this year—nearly a 50 per cent. increase on last year's sum of 27,000,000,000 roubles.—*Manchester Guardian Weekly*, June 2, 1939. [Per Mr. J. W. Ashley Smith.]

#### **1294. PREDICTION.**

The following is a rough translation of what appeared in a Polish newspaper :

#### HITLER, 1939?

	Wilhelm			
	Kaiser	Masaryk	Benes	Hitler
Born	1859	1850	1884	1889
Reigned from	1888	1918	1935	1933
Term	30 yrs.	17	3	6
Terminated age	59	85	54	50
Total	3836	3870	3876	3878
Divided by two	1918	1935	1938	1939
	Abdicated	Died	Resigned	???

—*Cavalcade*, May 13, 1939. [Per Prof. R. O. Street.]

## CLOCKS, CAMERAS, AND THE LORENTZ CONTRACTION.

BY REV. J. RIVERSDALE COLTHURST.

IN a previous paper, which appeared in *The Mathematical Gazette* for December 1938, the writer showed that, assuming the concept of a rigid rod, it is possible to derive the fundamental formulae of special Relativity from the consideration of photographs taken by two observers in relative uniform motion.

The extension of this method of approach to the consideration of the relativity of time measures appears to furnish some interesting results.



We consider two observers,  $X$  and  $Y$ , situated at  $O$  and  $O'$  respectively, who are receding from each other with uniform relative velocity  $v$ . Each possesses a clock, in the ordinary sense of the word; and these clocks are assumed to be synchronized, so that each marks zero at the coincidence of  $O$  and  $O'$ . The concept of a clock marking units of time is here assumed.

$X$  and  $Y$  part company at epoch zero as marked on each clock. Suppose  $X$  to take a photograph of  $Y$  at epoch  $t$  by his own clock, what will be the time  $\tau$  shown on  $Y$ 's clock in the photograph?

To determine this, let us make the further assumption that a clock marking units of time  $S$  to the observer connected with it will appear to mark units of time  $M$  to an observer moving with uniform velocity  $v$  relative to it.

Now at epoch  $t$  by his own clock  $X$  estimates the distance  $OO'$  as  $vt$ , and therefore estimates that a clock, at this distance, if relatively stationary, would show time  $t - vt/c$  in the photograph. But the clock actually shows a time  $\tau$ ; whence  $X$  derives the equation

$$\left(t - \frac{vt}{c}\right) \cdot S\text{-units} = \tau \cdot M\text{-units}.$$

It should be noted that a clock rigidly connected with  $O$ , and apparently coincident with  $O'$  in the photograph, will show the time  $t - vt/c$ .

Again, at epoch  $\tau$  by his own clock,  $Y$  estimates the distance  $O'O$  as  $v\tau$ , and so estimates that a ray of light leaving  $O'$  at this epoch will reach  $O$  at time  $\tau + v\tau/c$ . He is informed that it has reached  $O$  at time  $t$  by  $X$ 's clock; and he therefore obtains the equation

$$\left(\tau + \frac{v\tau}{c}\right) \cdot S\text{-units} = t \cdot M\text{-units}.$$

From these two equations we get

$$\tau \left(1 + \frac{v}{c}\right) / t = t \left(1 - \frac{v}{c}\right) / \tau \quad \text{or} \quad \tau = t \sqrt{\frac{1 - v/c}{1 + v/c}},$$

the correct relativity result. This result is reciprocal; for if  $Y$  now takes a photograph at time  $\tau_1$  by his own clock, showing  $X$ 's clock marking time  $t$ , we get similarly

$$t = \tau_1 \sqrt{\frac{1 - v/c}{1 + v/c}}.$$

If  $O$  and  $O'$  are approaching each other with uniform velocity  $v$ , and  $t$ ,  $\tau$ , and  $\tau_1$  have the same meaning, we find in this case that

$$\tau = t \sqrt{\frac{1 + v/c}{1 - v/c}} \quad \text{and} \quad t = \tau_1 \sqrt{\frac{1 + v/c}{1 - v/c}}.$$

Suppose now that each observer sends out light signals at intervals  $d\tau_0$  by his own clock, and that these are received by the other observer at intervals  $dt_0$  by his own clock, we get for recession

$$d\tau_0 = dt_0 \sqrt{\frac{1 - v/c}{1 + v/c}},$$

and for approach

$$d\tau_0 = dt_0 \sqrt{\frac{1 + v/c}{1 - v/c}},$$

which is of course the Doppler effect as regards time. For the case of recession each observer will perceive the course of events occurring at the other observer as though depicted in a slow motion film; while for approach they will appear as in a rapid motion film.

The Doppler effect then, of apparent retardation and acceleration of time, is capable of direct observation. A clock provided with a seconds pendulum, if approaching us, would show us the pendulum apparently oscillating faster than normal, while after it passed us it would be seen by us to be swinging more slowly than normal.

It is an interesting speculation that if a nebula, a million light years distant from us, had suddenly begun to approach us a million years ago with a velocity slightly less than that of light, then, given sufficient power of sight, it would be possible for us to see the happenings of that million years in the nebula crowded into, say, one hour of our time. To use the somewhat obscure phraseology of semi-popular Relativity, time would seem to be going tremendously fast in the nebula.

So far our results have been concerned only with directly observable effects. To pass from these to what is known as the Lorentz Contraction, let dashes refer to number measures applied by "stationary" observers to times as shown on "moving" clocks.

This implies the use by the "stationary" observer of his own unit to measure "moving" time.

From  $X$ 's point of view we now get  $\tau' . S\text{-units} = \tau . M\text{-units}$  and from  $Y$ 's point of view  $t' . S\text{-units} = t . M\text{-units}$ .

Using the previous equations, we get

$$\frac{\tau'}{\tau} = \frac{\tau(1 + v/c)}{t},$$

and also

$$\tau' = t(1 - v/c), \quad t' = \tau(1 + v/c),$$

whence eliminating  $t$  or  $\tau$  we get

$$\tau' = \tau \sqrt{\left(1 - \frac{v^2}{c^2}\right)} \quad \text{or} \quad t' = t \sqrt{\left(1 - \frac{v^2}{c^2}\right)},$$

the well-known formula for the "Lorentz Contraction". Here  $\tau'$  is the number measure attached by  $X$  to an interval of time measured as  $\tau$  by a clock moving relatively to himself with uniform velocity  $v$ ; and so for  $t'$ .

On putting  $t' \equiv f(t)$  and  $vt/c \equiv \phi(t)$ , and similarly for  $\tau$ , the two original equations take the form

$$f(t) = \tau + \phi(\tau),$$

$$f(\tau) = t - \phi(t);$$

and so Milne's fundamental equations, referred to in the previous paper, are reproduced, save that here  $t$  and  $\tau$  are directly observed by the same observer.

Again, let a light ray leave  $O'$  at local time  $\tau$  as before, and after reflection at  $O$  at local time  $t$ , return to  $O'$  at local time  $\tau_1$ ; or to put it otherwise, let  $Y$  take a photograph at his own time  $\tau_1$ , showing  $X$ 's clock marking time  $t$ .

Then

$$\tau = t \sqrt{\left(\frac{1 - v/c}{1 + v/c}\right)} \quad \text{and} \quad t = \tau_1 \sqrt{\left(\frac{1 - v/c}{1 + v/c}\right)},$$

whence

$$\begin{aligned} \frac{1}{2}(\tau + \tau_1) &= \frac{1}{2}t \left\{ \sqrt{\frac{1 - v/c}{1 + v/c}} + \sqrt{\frac{1 + v/c}{1 - v/c}} \right\} \\ &= t \sqrt{\left(1 - \frac{v^2}{c^2}\right)}, \end{aligned}$$

and it will be noticed that this result is the same for approach as for recession, since for the former we have merely to reverse the order of the terms inside the bracket.

If now  $Y$  assigns  $\tau_0 = \frac{1}{2}(\tau + \tau_1)$  as the epoch of the event of the reflection of the ray at  $O$ , we get  $t = \tau_0 \sqrt{\left(1 - \frac{v^2}{c^2}\right)}$ ; or the time measure of the epoch of an event occurring at  $O$ , as measured by  $X$ , is always less than the time measure of such event as calculated by  $Y$ ; and this irrespective of whether the motion is one of recession or approach. This has been described as Einstein's fundamental discovery.

The formulae  $t' = t \sqrt{\left(1 - \frac{v^2}{c^2}\right)}$  and  $t = \tau_0 \sqrt{\left(1 - \frac{v^2}{c^2}\right)}$  exhibit the Lorentz Contraction under two aspects. In each case  $t$  is directly measured by  $X$ : in the first case,  $t'$  is a number measure mentally assigned by  $Y$ , in order to reconcile the difference in units; while

in the second,  $\tau_0$  is a number measure calculated by  $Y$  from direct observations of his own.

A conceptual element thus makes its appearance in the Lorentz Contraction, regarded from either standpoint.

As regards time then, the Lorentz effect is always one of retardation; as regards space, one of contraction. The Doppler effects for both time and space differ for the cases of recession and approach; causing an apparent retardation or contraction in the one case, and an apparent acceleration or expansion in the other.

This peculiarity of the Lorentz effect seems, curiously enough, as regards its formulation

$$t = \tau_0 \sqrt{1 - \frac{v^2}{c^2}},$$

to depend on the fact that, if  $x$  be any positive number, then

$$\frac{1}{2}(x + 1/x) \geq 1;$$

the equality only holding good for  $x = 1$ . Putting

$$x = \sqrt{\frac{1 - v/c}{1 + v/c}},$$

we have  $\frac{1}{2}(x + 1/x) > 1$ , except for  $v = 0$ .

One of the famous paradoxes of Special Relativity seems to arise from neglecting the conceptual element in this formula.

The paradox is concerned with two observers, equipped with similar clocks, one of whom starts off on a voyage through space with a velocity nearly that of light. He is supposed to spend a year or so on his journey, and yet to discover, on rejoining his companion, that a hundred years have elapsed by the latter's clock. This paradox has engaged the attention of Langevin, Barnes, Eddington, and other writers.

An endeavour has been made to explain the paradox by drawing a distinction between the time of consciousness and the time of physical measurements. Barnes winds up his discussion of the question by saying: "Why the time of consciousness should go so much slower for one man than for the other has not, I think, been satisfactorily explained."

I venture to suggest that there may be nothing to explain!

In the first place, the formula  $t = \tau_0 \sqrt{1 - v^2/c^2}$  only applies to two observers in uniform relative motion, who have parted company at zero time by both clocks. As they are receding from each other in a straight line, they can never meet again, unless the geometry of space be elliptic!

Suppose, however, that their relative motion is not restricted to one of uniform velocity; but that  $\tau$ ,  $t$ ,  $\tau_1$ , and  $\tau_0$  have the same significance as previously; denoting the times of emission, reflection, and reception of a light signal; and the calculated epoch of reflection.

Then as before  $\tau_0 = \frac{1}{2}(\tau + \tau_1)$ ; and suppose that in general

$\frac{1}{2}(\tau + \tau_1)$  is some function of  $t$ , say  $\phi(t)$ . Then  $Y$  estimates  $\tau_0 = \phi(t)$ ; and if the two observers again coincide at epoch  $\tau$  by  $Y$ 's clock,  $\tau_1$  and therefore  $\tau_0$  will both be equal to  $\tau$ , since emission and reception are simultaneous.

We then get  $\tau = \phi(t)$  and similarly  $t = \phi(\tau)$ .

For the case of uniform relative velocity where  $\phi(t) \equiv t / \left(1 - \frac{v^2}{c^2}\right)$ , these equations can only be satisfied if  $v=0$  and hence  $t=\tau$ ; so that if the companions ever did meet again, they would find that their clock readings showed the time elapsed since last they met as being the same for both. The paradox has disappeared!

This result seems to be in full accord with that obtained for the general case by Milne and Whitrow in an article in the *Zeitschrift für Astrophysik*, which, they say, completely disposes of the "notorious clock paradox" of Special Relativity.

The aim of this, and the writer's previous paper, has been to indicate how two observers, equipped with clocks, cameras and rigid rods, can by direct observations formulate relations between number measures of length and time as read off on their own instruments; and to exhibit the Lorentz formulae as derivable therefrom by the injection of a conceptual element, or mental construct, typified by the number measures  $l'$ ,  $t'$ ,  $\tau'$ , and  $\tau_0$  used in the papers.

This position would seem to be substantiated by Broad when he declares that "The Lorentz Contraction ceases to be a physical shortening and becomes a question of units of measurement".

As in the previous paper the supposed "change of length" of the moving rod was shown to arise from the simple fact that the two observers, at the instant of coincidence, could take two different photographs of the same system; so in this paper the so-called "slowing down of time", or "change of rhythm" in the moving clock, is shown to be due to the fact that a photograph taken by one observer of two synchronized clocks which appear in the photograph to be at the same point, and one of which is moving relatively to him, will yet exhibit different times on the faces of the two clocks.

Both these surprising phenomena are dependent on nothing more subtle or abstruse than the finite velocity of light.

The writer must conclude by expressing his indebtedness to Dr. A. J. McConnell, F.T.C.D., Professor of Natural Philosophy in Dublin University, for many helpful discussions; and to Professor E. A. Milne for his course of lectures at the St. Andrews Colloquium of 1934, which first brought him to grips with the subject, and for some kindly words of encouragement since.

J. RIVERSDALE COLTHURST.

1295. Palmerston wrote gleefully that "this event is decisive of the ascendancy of Liberal Principles throughout Europe. . . . The reign of Metternich over the days of the Duke's policy might be measured by algebra, if not by arithmetic.—Phillip Guedalla, *The Duke*, p. 394. [Per Mr. M. W. Brown.]

# TRIANGLES WITH SIDES AND MEDIANS COMMENSURABLE.

By C. H. HARDINGHAM.

It is economical in the use of symbols to express the sides of a triangle fractionally, by dividing the expressions for  $a, b, c$  given in my previous Note (*Gazette*, XXIII, October, 1939, Note 1398, p. 380) by  $qy$ . Thus writing  $h$  for  $x/y$  and  $k$  for  $p/q$ , we have

$$a = hk + h - k + 1,$$

$$b = hk - h + k + 1,$$

$$c = 2(h + k),$$

and

$$n = hk - 1,$$

where  $l, m, n$  are the medians bisecting  $a, b, c$ .

If we now take

$$h = (2 - 5t - 6t^2)/(3 + 5t + t^2)$$

and

$$k = t(3 - 5t - 4t^2)/(4 + 5t + 3t^2),$$

the medians  $l$  and  $m$  will be commensurable. For instance, if  $t = -3/2$ , we have  $h = 16/9$ ,  $k = -9/13$ , and the sides are 262, 316, 254 and the medians 255, 204, 261.

These rather crooked expressions were formed in the following rather crooked manner. As in Note 1398, we have

$$4l^2 = (hk - 3h + 3k + 1)^2 + 32hk,$$

$$4m^2 = (hk + 3h - 3k + 1)^2 + 32hk,$$

and the equations to make  $l$  and  $m$  rational are

$$hk - 3h + 3k + 1 = \frac{8hk}{\sigma} - \sigma, \dots\dots\dots(i)$$

$$hk + 3h - 3k + 1 = \frac{8hk}{\rho} - \rho, \dots\dots\dots(ii)$$

giving  $2l = 8hk/\sigma + \sigma$ ,  $2m = 8hk/\rho + \rho$ .

If now we know one triangle  $a, b, c, l, m, n$ , we can derive a number of others from it, by first finding  $h'$  and  $k'$ , the values for this triangle of  $h$  and  $k$ , namely :

$$h' = (c + a - b)/(a + b - 2n), \quad k' = (c - a + b)/(a + b - 2n).$$

Next, putting these values in the equations (i) and (ii), we have quadratics for  $\sigma$  and  $\rho$  with roots  $s_1, s_2$  and  $r_1, r_2$ . Thus substituting either  $s$  for  $\sigma$  and either  $r$  for  $\rho$  in the equations and eliminating  $h$ , we have a quadratic for  $k$ , namely :

$$(3k + 1 + \sigma) \left/ \left( \frac{8k}{s} - k + 3 \right) \right. = h = (-3k + 1 + r) \left/ \left( \frac{8k}{r} - k - 3 \right) \right.$$

One root will be  $k'$ ; the other is a disappointment, being  $-h'$  giving  $h = -k'$ , since the equations (i) and (ii) are unaltered if  $-k, -h$

replace  $h, k$ . So, too, if we substitute  $sk/k', rk/k'$  for  $\sigma$  and  $\rho$ , giving

$$hk - 3h + 3k + 1 = 8hk'/s \quad - \quad sk/k', \dots\dots\dots(iii)$$

$$hk + 3h - 3k + 1 = 8hk'/r \quad - \quad rk/k', \dots\dots\dots(iv)$$

we find  $h', k'$  and  $1/k', 1/h'$  as the solutions, since (iii) and (iv) are unaltered if  $1/k, 1/h$  replace  $h, k$  and we multiply through by  $hk$ .

Thus in these eight cases no fresh triangle is found, but the eight substitutions  $s, rk/k'; sk/k', r$  give eight quadratics of which in each case one solution is  $h', k'$  and the other a fresh pair of values for  $h, k$ . The first gives, writing  $r'$  for  $r/k'$ ,

$$h = \frac{3 + (3 - r')(1 + s)}{3 + (3 - \frac{8}{r})(1 - \frac{8}{s})} \cdot \frac{1}{h'}, \quad k = \frac{3 + (3 - \frac{8}{r})(1 + s)}{3 + (3 - r')(1 - \frac{8}{s})} \cdot \frac{1}{k'}.$$

Thus for the triangle of sides 254, 262, 316, we have  $k=3, h=77/27, \sigma=14/3$  or  $-44/3, \rho=-14$  or  $44/9$ , and we have the following table of values :

	$\sigma$	$\rho$	$k$	$h$
1.	$\frac{14}{3}$	$-\frac{14k}{3}$	-4	$-\frac{133}{3}$
2.	$\frac{14}{3}$	$\frac{44k}{27}$	$-\frac{2709}{2101}$	$\frac{109}{126}$
3.	$-\frac{44}{3}$	$-\frac{14k}{3}$	$\frac{473}{343}$	$-\frac{229}{66}$
4.	$\frac{44}{3}$	$\frac{44k}{27}$	$\frac{36}{19}$	$-\frac{1001}{9}$
5.	$\frac{14k}{9}$	-14	$-\frac{27}{8}$	$-\frac{161}{129}$
6.	$\frac{14k}{9}$	$\frac{44}{9}$	$\frac{1177}{7}$	$-\frac{302}{63}$
7.	$-\frac{44k}{9}$	-14	$-\frac{1701}{11}$	$\frac{62}{33}$
8.	$-\frac{44k}{9}$	$\frac{44}{9}$	$\frac{7}{8}$	$-\frac{517}{387}$

Of these, 4, 6 and 7 fail to give real triangles. From any one of these we can start afresh and find eight more triangles, or, reverting to the original triangle, interchange  $c$  with  $a$  or  $b$ , and so obtain two fresh sets of eight triangles which turn out to be different from those already found. We can also put  $-b$  for  $b$ , or we can take the triangle whose sides are two-thirds of the medians and whose medians are half the sides. In this case we obtain sides 170, 174, 136 with medians 131, 127, 158, which are the smallest numbers I have discovered.

If we put  $h=1$ ,  $k=t$ , we have the flat triangle  $a=2$ ,  $b=2t$ ,  $c=2t+2$ ,  $l=t+2$ ,  $m=2t+1$ ,  $n=t-1$ , and we find  $\sigma=2$  or  $-4t$ ,  $\rho=-4$  or  $2t$ . Of the eight triangles derived from these values all are flat but those given by  $\sigma=2k/t$ ,  $\rho=2t$  and  $\sigma=-4t$ ,  $\rho=-4k/t$ . The latter gives the values for  $h$  and  $k$  which were stated above. The former changes  $h$ ,  $k$  and  $t$  to their reciprocals. Cleared of fractions, the sides and medians are

$$\begin{aligned} a &= 20 + 22t - 30t^2 + 8t^3 + 60t^4 + 28t^5, \\ b &= 4 + 60t + 56t^2 + 30t^3 + 46t^4 + 20t^5, \\ c &= 16 - 2t - 86t^2 - 158t^3 - 86t^4 - 8t^5, \\ l &= 6 - t + 69t^2 + 126t^3 + 64t^4 + 6t^5, \\ m &= 18 + 8t - 28t^2 - 93t^3 - 53t^4 - 18t^5, \\ n &= -12 - 29t - 63t^2 - 21t^3 + 47t^4 + 24t^5. \end{aligned}$$

In making trials it is well to observe that the conditions for a true triangle are that if  $h$  and  $k$  have the same sign, then  $|h|$  and  $|k|$  must be both greater or both less than unity, while if  $h$  and  $k$  have different signs, one of  $|h|$ ,  $|k|$  must be greater and the other less than unity.

C. H. H.

**1296.** The certainty of intuition in which we say that some particular piece of music is beautiful is unanswerable and immediate and final in a degree quite impossible to the demonstrations addressed to our reason by the scientist or philosopher. Even the certainties of pure mathematics, our assurance that two and two make four, or that the angles at the base of an isosceles triangle are equal, have quite another directness and force than our intuitions of beauty, or of any kind of value, and are more open to challenge.—Prof. L. W. Grensted, *This Business of Living*. [Per Mr. A. F. Mackenzie.]

**1297.** . . . We recognize that human personality is richer and more complete in proportion to the fulness and range of its appreciation of values. . . . But this recognition of values is the recognition of values to be expressed or attained, and once more we know, with a very varying sense of its actual urgency for ourselves, that this recognition involves us at once in freedom to seek their attainment and in moral responsibility for our use of that freedom. Freedom in this personal sense is the despair of the scientist. He can neither analyse it, nor give any account of its origin and development. His very methods, except when they are purely descriptive, leave no place for it, since they depend upon the application of strict rules of cause and effect, or of the mathematical equations which are their modern substitute. But neither causality nor mathematical equations allow for freedom. The best that the latter can do is to make some place for indeterminism, on the principle that when an equation is satisfied by two solutions, you may choose either of them. But this indeterminism has nothing in common with the freedom which, in this case, would choose one solution rather than the other on the ground of some inherent and recognized value.—Prof. L. W. Grensted, *This Business of Living*. [Per Mr. A. F. Mackenzie.]

**1298.** The average school graduate has a vague notion that 70% is passing and may take years to learn that in adding a column of figures the passing mark is 100%, and in marking a bill of material a mistake in the decimal point which cost \$10,000 is a pretty bad error.—David Cushman Coyle, in *School Life*. [Per Mr. E. B. Escott.]

By C. G. PARADINE.

From the equation  $AB^2 + AC^2 = 2AD^2 + 2BD^2$  it follows that if  $AB$ ,  $AC$  and  $AD$  are to be integral, then  $2BD^2$  is an integer, whence, if  $BC$  is to be integral,  $BD$  is an integer. The bisected side is therefore even. We require all solutions in integers of the equation

where the sides are  $2a$ ,  $2b$  and  $2c$  and the median is  $x$ , except that when  $a$  and  $x$  are even there is a triangle of sides  $a$ ,  $b$ ,  $c$  and median  $\frac{1}{2}x$ . Alternatively each integral solution of this equation yields a parallelogram having integral sides  $b$ ,  $c$  and integral diagonals  $a$ ,  $x$ . When the parallelogram is a rectangle,  $x=a$ , so that Pythagorean numbers appear as a special case of Apollonian numbers.

Figure 1 shows a graph of the function  $a = 5 - x^2$ . The vertical axis is labeled  $a$  and has tick marks from 0 to 5. The horizontal axis is labeled  $x$  and has tick marks from 0 to 5. The curve is a parabola opening downwards, starting at  $(0, 5)$  and ending at  $(5, 0)$ . The area under the curve is divided into five vertical strips of width 1 unit each. The heights of the strips are labeled at the top: 50, 41, 34, 29, and 25. The x-coordinates of the strip boundaries are labeled at the bottom: 0, 1, 2, 3, 4, and 5. The formula  $(a^2 + x^2)$  is written below the x-axis.

Figure 1 shows a graph of the function  $a = \frac{1}{2}(a^2 + x^2)$  for  $a \in [0, 5]$  and  $x \in [0, 5]$ . The horizontal axis is labeled  $x$  and the vertical axis is labeled  $a$ . The graph consists of a series of points forming a parabolic shape, with the following coordinates: (0,0), (1,1), (2,4), (3,9), (4,16), and (5,25). The points are plotted at integer intervals for both  $x$  and  $a$ .

$$b^2 + c^2 = \frac{1}{2}\{(b+c)^2 + (b-c)^2\},$$

which yields two degenerate triangles as solutions of our problem. If a number which is the sum of two squares occurs only once in our table, it can yield no other solution. If, however, it occurs more than once, so that  $b_1^2 + c_1^2 = b_2^2 + c_2^2$ , then two real triangles are obtained by adopting as the values of  $b, c$  the more nearly equal pair.

and as  $a$ ,  $x$  the sum and difference of the other pair. The implied inequalities are

(i)  $b_1 + c_1 > (b_2 + c_2)$ , (ii)  $b_1 + (b_2 + c_2) > c_1$ , (iii)  $c_1 + (b_2 + c_2) > b_1$ , and

(iv)  $b_1 + c_1 > (b_2 - c_2)$ , (v)  $b_1 + (b_2 - c_2) > c_1$ , (vi)  $c_1 + (b_2 - c_2) > b_1$ .

By hypothesis  $b_1 > c_1$ ,  $b_2 > c_2$  and  $b_1 < b_2$ . From these (ii), (iii) and (v) follow immediately, and (iv) after squaring. To prove (i) and (vi) we may set  $b_1^2 + c_1^2 = b_2^2 + c_2^2 = r^2$ :

$$b_1 = r \cos \theta_1, \quad c_1 = r \sin \theta_1, \quad b_2 = r \cos \theta_2, \quad c_2 = r \sin \theta_2,$$

where now  $45^\circ > \theta_1 > \theta_2$ , and we have to show that

$$\cos \theta_1 + \sin \theta_1 > \cos \theta_2 + \sin \theta_2$$

and

$$\cos \theta_2 - \sin \theta_2 > \cos \theta_1 - \sin \theta_1;$$

but these are both equivalent, by squaring (which is legitimate), to

$$\sin 2\theta_1 > \sin 2\theta_2,$$

and this is true.

Thus, since  $65 = 8^2 + 1^2 = 7^2 + 4^2$ , we may deduce the equation  $7^2 + 4^2 = \frac{1}{2}(9^2 + 7^2)$ , which in turn yields the triangles in which

$$2a = 18, \quad 2b = 14, \quad 2c = 8, \quad x = 7;$$

$$2a = 14, \quad 2b = 14, \quad 2c = 8, \quad x = 9.$$

The second of these is isosceles and has two integral medians.

3. Let  $N = b^2 + c^2$ . If  $b$ ,  $c$  have a common factor, it is a factor of  $b + c$  and  $b - c$ , and its square is a factor of  $N$ . Such a value of  $N$  can only yield a solution in which  $b$ ,  $c$ ,  $a$ ,  $x$  have a factor in common. (It is possible, however, that  $b_1$ ,  $c_1$  have a common factor, but not  $b_2$ ,  $c_2$ .) If  $b$ ,  $c$  are prime to one another, then  $N$ , and so also each of its prime factors, has  $-1$  as a quadratic residue. For by the usual method of continued fractions we can find  $h$ ,  $K$  such that

$$bh - NK = +1 \text{ or } -1,$$

i.e. mod  $N$ ,

$$bh \equiv +1 \text{ or } -1,$$

or

$$b^2h^2 \equiv 1.$$

But  $c^2 \equiv -b^2$ , and hence  $c^2h^2 \equiv -b^2h^2 \equiv -1$ , so that  $-1$  is the residue of  $(ch)^2$ , mod  $N$ .

The primes of which  $-1$  is a quadratic residue are 2 and those of the form  $4n + 1$ . Any such prime can be expressed as the sum of two squares in one way only. An odd number which is the product of primes of the form  $4n + 1$  can be expressed as the sum of two squares in as many ways as it can be expressed as the product of two factors, including, as one way, the product of itself and 1. We shall construct all fundamental solutions of our equation if we construct all composite numbers of the form  $5^a \cdot 13^b \cdot 17^c \cdot 29^d \dots$  and their doubles, express them in all possible ways in the form  $b^2 + c^2$ , and adopt every pair  $b$ ,  $c$  in conjunction with the sum and difference of a more unequal pair belonging to the same  $N$ , as a

solution to be denoted as  $b, c; a, x$ . It is sufficient to apply the process to odd values of  $N$ ; solutions corresponding to  $2N$  can then be written down by virtue of the relations :

$$N = b_1^2 + c_1^2 = \frac{1}{2}\{(b_2 + c_2)^2 + (b_2 - c_2)^2\},$$

$$2N = (b_2 + c_2)^2 + (b_2 - c_2)^2 = \frac{1}{2}\{(2b_1)^2 + (2c_1)^2\}.$$

The necessity for the inclusion of the exponents  $\alpha, \beta, \dots$  is related to the possibility, referred to above, that  $b_2, c_2$  are mutually prime, though  $b_1, c_1$  are not; for example  $5^3 = 11^2 + 2^2 = 10^2 + 5^2$ , yielding the solution 10, 5; 13, 9.

4. The work of expressing  $N$  as the sum of two squares is most easily done in practice by subtracting from  $N$  the perfect squares between  $N$  and  $\frac{1}{2}N$  and inspecting the remainders. There is, however, a direct theoretical process if the primes concerned are known in the required form.

If  $N = (p^2 + q^2)(r^2 + s^2)$ ,  
 then  $N = (pr + qs)^2 + (ps - qr)^2$   
 $= (pr - qs)^2 + (ps + qr)^2.$

Further, a prime (or other number)  $d$  can be expressed as the sum of two squares if we know  $n$ , such that  $n^2 \equiv -1, \text{ mod } d$ . The method is given in Caliban's *Problem Book*, where acknowledgment is made to Professor Jolliffe and is as follows: in the division process for the H.C.F. (which is 1) of  $d$  and  $n$ , if the first two consecutive remainders less than  $\sqrt{d}$  are  $p$  and  $q$ , then  $d = p^2 + q^2$ . Conversely, as in § 2 above, if  $d$  is known as the sum of two squares,  $n$  can be found.

As an illustration, consider the number 1105. Subtracting in turn the squares 1089, 1024, 961, 900, 841, 784, 729, 676, 625 and 576, we find  $1105 = 33^2 + 4^2 = 32^2 + 9^2 = 31^2 + 12^2 = 24^2 + 23^2$ . These equations could have been obtained as follows:

$$1105 = 5 \cdot 13 \cdot 17.$$

But  $5 = 2^2 + 1^2 = (2, 1)$ , say, and  $13 = (3, 2)$ .

Thus  $65 = (8, 1)$  or  $(7, 4)$  by application of the formulae in  $p, q, r, s$  above.

Then  $65 \times 17 = (8, 1) \times (4, 1) = (33, 4)$  or  $(31, 12)$ ,  
 and  $= (7, 4) \times (4, 1) = (32, 9)$  or  $(24, 23)$ .

Tabulate the pairs  $b, c$  with their sums and differences:

$$\begin{array}{l} 33, 4 : 37, 29 \\ 32, 9 : 41, 23 \\ 31, 12 : 43, 19 \\ 24, 23 : 47, 1. \end{array}$$

Any left-hand pair may be associated with a higher right-hand pair, and we have, for possible values of  $b, c; a, x$ :

$$\begin{array}{l} 32, 9 : 37, 29 \\ 31, 12 : 37, 29 \end{array}$$

$$31, 12 : 41, 23$$

$$24, 23 : 37, 29$$

$$24, 23 : 41, 23$$

$$24, 23 : 43, 19,$$

which by the interchange of  $a$  and  $x$  yield 12 triangles.

5. I have constructed all composite numbers of the relevant form less than 5000 and tabulated about 1000 triangles. Of these, some 40 have two integral medians, but so far I have found no triangle having all its sides and medians integral. These numbers do not include triangles whose sides have a common factor nor, as having two medians, isosceles triangles with integral medians to the equal sides. To facilitate the search for composite figures, results were retabulated to show against values of  $a$  all pairs  $b, c$ . This second table presents a somewhat elusive pattern which might be capable of expression in a single formula. By inspection I have constructed a formula which gives all the solutions in the early stages but later begins to be guilty of a few omissions. Moreover, it does not incorporate the conditions for the triangle to be real, though it is easy to cease operations at the points where the triangle becomes imaginary.

Let the bisected side  $= 2a$ .

Let  $b = na - (2n^2 - 1)p$  numerically,

and  $c = (n+1)a - \{2(n+1)^2 - 1\}p$  numerically.

Then  $x = (2n+1)a - \{(2n+1)^2 + 1\}p$  numerically,

where  $n, p$  are positive integers.

Thus  $b, c$  are consecutive members of a quadratic sequence. *E.g.* when  $a=13$  and  $p=1$  we obtain by putting  $n=1, 2, 3 \dots$  the sequence :

		12	19	22	21	16	7	-6	.
and when	$p=2$	11	12	5	-10	-33	.	.	.
	$p=3$	10	5	-12	.	.	.	.	.
	$p=4$	9	-2	.	.	.	.	.	.
	$p=5$	8	-9	.	.	.	.	.	.

In this case we have the complete solution, as found by the method of §§ 3, 4; that is, when  $a=13$  the pair  $b, c$  can be 12, 19; 19, 22; 22, 21; 21, 16; 16, 7; 11, 12; 12, 5; 10, 5 or 8, 9.

The first omission by the formula is the solution  $a=16, b=c=17$ . For values of  $a$  up to 25 there are 10 omissions out of about 200 solutions.

6. The problem of finding a triangle having all its sides and medians integral has, in the foregoing, been regarded as a special case of the simpler one, which we may say has been solved, of ensuring integral sides and one integral median. It is more attractive to retain symmetry, and there are several symmetrical forms in which the question may be posed.

(i) Find three positive integers  $a, b, c$  capable of forming the sides of a real triangle such that  $2b^2 + 2c^2 - a^2$  and the two similar expressions are perfect squares.

(ii) We may ensure that the triangle is real by setting  $b + c - a = l$ , etc. We have then to find three positive integers  $l, m, n$  such that  $4l^2 + m^2 + n^2 - 2mn + 4nl + 4lm$  and the two similar expressions are each the square of an even number.

(iii) If we put  $b^2 + c^2 - a^2 = p$ , etc., so that  $q + r = 2a^2$ , etc., we require to find integers  $p, q, r$  such that  $4p + q + r$  and the two similar expressions are each twice a perfect square. One of  $p, q, r$  can be negative, but their sums two at a time must all be positive. We can easily show that  $p, q, r$  must all be even. For

$$2x^2 - 2a^2 = (4p + q + r) - (q + r),$$

whence  $x^2 - a^2 = 2p$ .

Hence 2 is a factor of  $x - a$  or  $x + a$ . But if it is a factor of the one it is a factor of the other, and so also of  $p$ . (As a special case it is possible that  $x = a$  and  $p = 0$ , which may be included as even.)

(iv) From the equations

$$2(b^2 + c^2) = a^2 + x^2,$$

$$2(c^2 + a^2) = b^2 + y^2,$$

$$2(a^2 + b^2) = c^2 + z^2,$$

we may deduce

$$2(y^2 + z^2) = x^2 + 9a^2$$

and two similar equations, showing that  $3a, 3b, 3c$  are the medians of the triangle whose sides are  $2x, 2y, 2z$ . To construct the second triangle from a given triangle  $ABC$ , complete the parallelogram  $ABXC$  and produce  $BC$  to  $Y$  so that  $BC = CY$ ; then  $AXY$  is the required triangle. Its area is clearly three times that of the parent triangle. This relation is interesting when deduced algebraically. If  $\Delta, \Delta_1$  denote the areas of the first and second triangles, then

$$\Delta^2 = 2\Sigma b^2c^2 - \Sigma a^4 \text{ and } \Delta_1^2 = 2\Sigma y^2z^2 - \Sigma x^4.$$

Using our initial equations we find not merely  $\Delta_1^2 = 9\Delta^2$  but  $\Sigma y^2z^2 = 9\Sigma b^2c^2$  and  $\Sigma x^4 = 9\Sigma a^4$ . The implied relation  $\Sigma x^2 = 3\Sigma a^2$  is, of course, evident from the initial equations. If we now evaluate  $x^2y^2z^2$  in terms of  $a^2, b^2$  and  $c^2$ , we may express the relationship between sides and medians by saying that if the squares of the halved sides are the roots of the equation

$$S^3 - pS^2 + qS - r = 0,$$

then the squares of the medians are the roots of

$$M^3 - 3pM^2 + 9qM + 4p^3 - 18pq + 27r = 0.$$

This result would seem to be useful for the purpose of integral solutions, only if we knew some convenient way of expressing the fact that the roots of a cubic equation are all the squares of integers.

C. G. PARADINE.

SOME THEOREMS CONNECTED WITH  
MACLAURIN'S INTEGRAL TEST.

BY C. T. RAJAGOPAL.

1. The theorem of J. E. Littlewood stated below \* includes Mac-laurin's integral test and furnishes the basis for a unified theory of convergence criteria for series of positive terms.†

THEOREM A. If  $F(x)$  is positive monotone decreasing and  $(D_n)$  is such that

$$\left. \begin{aligned} 0 = D_0 < D_1 < \dots < D_{n-1} < D_n \rightarrow \infty \\ d_n = D_n - D_{n-1} \leq k \text{ for } n \geq 1 \end{aligned} \right\} (n \rightarrow \infty),$$

then  $\sum d_n F(D_n)$  converges or diverges with  $\int_1^\infty F(x) dx$ .

The main object of this paper is to elaborate and extend Theorem A, indicating certain immediate applications of the resulting theorems. Incidentally, it establishes the existence of a constant

$$\gamma_d = \lim_{n \rightarrow \infty} \left( \sum_{v=1}^n \frac{d_v}{D_v} - \log D_n \right),$$

which has some of the more elementary properties and applications of Euler's constant  $\gamma$ .

2. A slight generalization of the proof of Maclaurin's integral test establishes the following theorem equivalent to Littlewood's.

THEOREM I. If  $F(x)$ ,  $(D_n)$  are as in Theorem A, then

$$\left[ \sum_{v=1}^n d_v F(D_v) - \int_{D_1}^{D_n} F(x) dx \right]$$

decreases steadily to a limit such that

$$(d_1 - k) F(D_1) \leq \lim_{n \rightarrow \infty} \left[ \sum_{v=1}^n d_v F(D_v) - \int_{D_1}^{D_n} F(x) dx \right] \leq d_1 F(D_1).$$

COROLLARY 1. Taking  $F(x) = \frac{1}{x}$ , we see that  $\left( \sum_{v=1}^n \frac{d_v}{D_v} - \log D_n \right)$

tends to a limit  $\gamma_d$  between  $\frac{d_1 - k}{D_1} - \log D_1$  and  $\frac{d_1}{D_1} - \log D_1$ .

COROLLARY 2. If  $F(x) = \frac{1}{x \cdot l_1 x \cdot l_2 x \dots l_p x}$ , where  $l_1 x = \log x$ ,  $l_2 x = \log \log x$ , etc., we obtain the result :

$$\lim_{n \rightarrow \infty} \left( \sum_{v=m}^n \frac{d_v}{D_v \cdot l_1 D_v \cdot l_2 D_v \dots l_p D_v} - l_{p+1} D_v \right)$$

is finite,  $m$  being so large that  $l_p D_m > 0$ .

\* Note on the convergence of series of positive terms, *Messenger of Maths.*, 39 (1910), 191-2.

† For a discussion of this point of view, vide C. T. Rajagopal, Convergence theorems for series of positive terms, *Journ. Indian Math. Soc.* (New Series), 3 (1938), 118-125.

*Example.* The following is an instance of a problem to which we may apply argument of the type which establishes Theorem I.

Let  $F(x)$  be as in the Theorem and let  $F(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Rearrange the terms of the convergent series  $s \equiv \sum (-1)^{n-1} F(n)$ , denoting by  $p$  and  $q$  the number of positive and negative terms respectively in any partial sum of the rearrangement. If

$$\frac{F(n) \cdot D_n}{D_n - D_{n-2}} \rightarrow g \quad \text{and} \quad \frac{D_{2p}}{D_{2q}} \rightarrow l,$$

then the sum of the rearrangement is  $s + g \log l$ .

A proof of this result is readily constructed on the lines of that of its particular case  $D_n = n$ .\*

The next two theorems are extensions of Theorem I; in them we drop the restrictions of monotony and reality imposed on  $F(x)$ , as Hardy does in his generalizations of Maclaurin's integral test.†

**THEOREM II.** *If  $F(x)$  has a continuous derivative  $F'(x)$  such that  $\int_0^\infty |F'(x)| dx$  is convergent and  $(D_n)$  is as in Theorem A, then*

$$\lim_{n \rightarrow \infty} \left[ \sum_{v=1}^n d_v F(D_v) - \int_{D_1}^{D_n} F(x) dx \right]$$

is finite.

*Proof.*

$$d_v F(D_v) = \int_{D_{v-1}}^{D_v} F(x) dx + \int_{D_{v-1}}^{D_v} \{F(D_v) - F(x)\} d(x - D_{v-1}).$$

Integrating by parts the second term of the right-hand member, we write

$$d_v F(D_v) = \int_{D_{v-1}}^{D_v} F(x) dx + \int_{D_{v-1}}^{D_v} (x - D_{v-1}) F'(x) dx.$$

Summing this equality from  $v=2$  to  $v=n$ , we get

$$\sum_{v=2}^n d_v F(D_v) - \int_{D_1}^{D_n} F(x) dx = \sum_{v=2}^n \int_{D_{v-1}}^{D_v} (x - D_{v-1}) F'(x) dx.$$

Since

$$|j_v| \equiv \left| \int_{D_{v-1}}^{D_v} (x - D_{v-1}) F'(x) dx \right| \leq k \int_{D_{v-1}}^{D_v} |F'(x)| dx,$$

the convergence of  $\sum j_v$  follows from that of  $\int_0^\infty |F'(x)| dx$ . This proves the existence of

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[ \sum_{v=1}^n d_v F(D_v) - \int_{D_1}^{D_n} F(x) dx \right] = S_\infty \quad (\text{say}).$$

\* Bromwich, *Infinite Series* (1926), 76.

† Theorems connected with Maclaurin's test for the convergence of series, *Proc. Lond. Math. Soc.* (2), 9 (1911), 126-144, Ths. 1 and 2.

## COROLLARY 1.

$$S_n - S_\infty = - \sum_{v=n}^{\infty} \int_{D_v}^{D_{v+1}} (x - D_v) F'(x) dx.$$

In the particular case  $F(x) = 1/x$ , the right-hand member of the last equality is  $O(1/D_n)$ ; whence

$$\left[ \sum_{v=1}^n \frac{d_v}{D_v} - \log D_n \right] - \gamma_d = O\left(\frac{1}{D_n}\right) \quad (n \rightarrow \infty).$$

COROLLARY 2. If  $\mu$  is complex and  $R\mu > 0$ , then the series  $\sum \frac{d_n}{D_n^\mu}$  is convergent when  $R\mu > 1$ ; divergent\* when  $R\mu < 1$ , the sum to  $n$  terms being  $\sim \frac{D_n^{1-\mu}}{1-\mu}$ . When  $R\mu = 1$ , the series oscillates finitely, the range of oscillation being  $\frac{2}{Im\mu}$  both for its real and imaginary parts.

This follows at once from Theorem II by taking  $F(x) = x^{-\mu}$ .†

When  $\int_0^\infty |F'(x)| dx$  is divergent, the following modification of Theorem II is sometimes useful.

THEOREM III. If  $F(x)$  has continuous derivatives of the first two orders,  $\int_0^\infty |F''(x)| dx$  is convergent, and  $(D_n)$  is as in Theorem A, then  $\lim_{n \rightarrow \infty} \left[ \sum_{v=2}^n \frac{d_v}{2} \{F(D_v) + F(D_{v-1})\} - \int_{D_1}^{D_n} F(x) dx \right]$  is finite.

*Proof.* Transforming the expression  $j_v$  (in the proof of Theorem II) by a further integration of parts, we obtain the identity

$$\begin{aligned} \int_{D_{v-1}}^{D_v} F(x) dx - d_v F(D_v) + \frac{d_v}{2} \int_{D_{v-1}}^{D_v} F'(x) dx \\ = \frac{1}{2} \int_{D_{v-1}}^{D_v} (x - D_{v-1})(x - D_v) F''(x) dx, \end{aligned}$$

or,

$$\begin{aligned} \frac{d_v}{2} \{F(D_v) + F(D_{v-1})\} - \int_{D_{v-1}}^{D_v} F(x) dx \\ = -\frac{d_v^2}{2} \int_{D_{v-1}}^{D_v} \phi_2\left(\frac{x - D_{v-1}}{d_v}\right) F''(x) dx, \end{aligned}$$

where  $\phi_2(x)$  denotes the second degree Bernoullian polynomial.

The desired result follows by summing the last identity from  $v = 2$  to  $v = n$  and letting  $n \rightarrow \infty$ .

\* The divergence of a complex series is defined by Bromwich (*op. cit.* 233) as the divergence of the sequence of moduli of the partial sums of the series.

† The particular case  $D_n = n$  is proved by Bromwich (*op. cit.* 235-7) by a method which in principle is not very different from the one employed here.

COROLLARY. Let  $F(x) = \log x$ . Then

$$\lim_{n \rightarrow \infty} \left[ \sum_{v=1}^n \frac{(d_v + d_{v+1})}{2} \log D_v - \frac{(D_n + D_{n+1})}{2} \log D_n + D_n \right] = \log C \quad (\text{say}),$$

$$\text{or,} \quad \prod_{v=1}^n D_v^{\frac{d_v + d_{v+1}}{2}} \sim C e^{-D_n} D_n^{\frac{D_n + D_{n+1}}{2}} \quad (n \rightarrow \infty),$$

which reduces to Stirling's asymptotic formula for  $n!$  when  $D_n = n$ . A result more precise than the last is evidently

$$\prod_{v=1}^n D_v^{\frac{d_v + d_{v+1}}{2}} \sim C e^{-D_n} D_n^{\frac{D_n + D_{n+1}}{2}} \left[ 1 + O\left(\frac{1}{D_n}\right) \right] \quad (n \rightarrow \infty).$$

3. Two well-known results involving  $\gamma$  are :

$$\text{THEOREM B.} \quad \lim_{\mu \rightarrow 1+0} \left( \sum_{n=1}^{\infty} \frac{1}{n^{\mu}} - \frac{1}{\mu-1} \right) = \gamma \quad [\text{DIRICHLET}];$$

$$\text{THEOREM C.} \quad 1 - \gamma = \sum_{n=2}^{\infty} \frac{s(n)}{n}, \quad \text{where } s(\kappa) = \sum_{v=2}^{\infty} \frac{1}{v^{\kappa}} \quad [\text{EULER}].$$

The two theorems which follow are extensions to  $\gamma_d$  of Theorems B and C.

$$\text{THEOREM IV.} \quad \lim_{\mu \rightarrow 1+0} \left[ \sum_{n=1}^{\infty} \frac{d_n}{D_n^{\mu}} - \frac{1}{(\mu-1)D_1^{\mu-1}} \right] = \gamma_d + \log D_1.$$

*Proof.* This is modelled on the proof of Theorem B.

If  $\mu > 1$ , we have, by Theorem I,

$$f(N) \equiv \sum_{n=1}^{\infty} \frac{d_n}{D_n^{\mu}} - \int_{D_N}^{\infty} \frac{dx}{x^{\mu}} \begin{cases} < \frac{d_N}{(D_N)^{\mu}} < \frac{d_N}{D_N} \\ > \frac{d_N - k}{(D_N)^{\mu}} \geq \frac{d_N - k}{D_N} \end{cases} \quad (D_N > 1),$$

so that  $f(N) \rightarrow 0$  (as  $N \rightarrow \infty$ ) uniformly for  $\mu \geq 1$ . The required limit is that of

$$f(1) \equiv \sum_{n=1}^{\infty} \frac{d_n}{D_n^{\mu}} - \int_{D_1}^{\infty} \frac{dx}{x^{\mu}} = \sum_{v=1}^{N-1} \frac{d_v}{D_v^{\mu}} - \int_{D_1}^{D_N} \frac{dx}{x^{\mu}} + f(N).$$

Hence

$$\lim_{\mu \rightarrow 1+0} f(1) = \left( \sum_{v=1}^N \frac{d_v}{D_v} - \log D_N \right) - \frac{d_N}{D_N} + \log D_1 + \lim_{\mu \rightarrow 1+0} f(N).$$

As  $N \rightarrow \infty$ , the right-hand member tends to  $\gamma_d + \log D_1$ . The left-hand member being independent of  $N$ , our theorem follows.

$$\text{THEOREM V.} \quad \text{If} \quad s_d(\kappa) = \sum_{v=2}^{\infty} \left( \frac{d_v}{D_v} \right)^{\kappa}, *$$

$$\text{then} \quad 1 - \gamma_d - \log D_1 = \sum_{n=2}^{\infty} \frac{s_d(n)}{n}.$$

\* It is obvious that this series is convergent for  $\kappa > 1$ .

*Proof.*

$$\sum_2^{\infty} \frac{s_d^{(n)}}{n} = \frac{1}{2} \left[ \left( \frac{d_2}{D_2} \right)^2 + \left( \frac{d_3}{D_3} \right)^2 + \dots \right] \\ + \frac{1}{3} \left[ \left( \frac{d_2}{D_2} \right)^3 + \left( \frac{d_3}{D_3} \right)^3 + \dots \right] \\ + \dots$$

Summing by columns the double series which forms the right-hand member, we find that

$$\sum_2^{\infty} \frac{s_d^{(n)}}{n} = \sum_{n=2}^{\infty} \left[ \frac{1}{2} \left( \frac{d_n}{D_n} \right)^2 + \frac{1}{3} \left( \frac{d_n}{D_n} \right)^3 + \dots \right] \\ = \sum_{n=2}^{\infty} \left[ \log \left( \frac{1}{1 - \frac{d_n}{D_n}} \right) - \frac{d_n}{D_n} \right] \\ = \lim_{n \rightarrow \infty} \left[ \sum_{v=2}^n \log \frac{D_v}{D_{v-1}} - \sum_{v=2}^n \frac{d_v}{D_v} \right] \\ = 1 - \log D_1 - \gamma_d.$$

*Examples.* By proceeding as in the proof of Theorem V, we can show that, when  $d_n = k$ ,

$$(i) \quad 1 - \gamma_d - \log \frac{D_1 + D_2}{2} = \sum_{n=1}^{\infty} \frac{s_d^{(2n+1)}}{(2n+1)2^{2n}}; \\ (ii) \quad 1 - \gamma_d - \log (D_1 \cdot D_2)^{\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{s_d^{(2n+1)}}{(2n+1)}.$$

In the particular case  $d_n = 1$ , these results give Legendre's well-known expressions for  $\gamma$ .

4. In conclusion I state two theorems which may be proved by the use of  $\gamma_d$ . The first of these contains as part a generalization of Gauss's test and the second a generalization of the analogue of Gauss's test for alternating series. The theorems can be proved in precisely the same way as their particular cases in which  $D_n = n$  discussed by me in a note in the *Gazette*.\*

**THEOREM VI.** If  $\Sigma a_n$  is a series of positive terms,  $(D_n)$  is a sequence defined as in Theorem A and

$$\text{or,} \quad \left. \begin{aligned} & \frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \\ & \frac{1}{d_n} (a_{n+1} \cdot d_n - 1) \end{aligned} \right\} = -\frac{\mu}{D_n} - O\left(\frac{1}{D_n^\lambda}\right) \quad (\lambda > 1),$$

\* *Math. Gazette*, 21 (1937), 161-3.

then either of the conditions: (i)  $\mu > 1$ , (ii)  $\lim_{n \rightarrow \infty} \frac{a_n D_n}{d_n} = 0$ ,\* is necessary and sufficient for the convergence of  $\Sigma a_n$ .

*Remark.* Elsewhere † I have shown that the Bertrand-de Morgan sequence of convergence criteria for series of positive terms can be expressed in a form analogous to that of Kummer's convergence criterion:

If  $\Sigma a_n$ ,  $(D_n)$  are as in Theorem VI, and if

$$\left. \begin{aligned} & \frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \\ \text{or, } & \frac{1}{d_n} \left( \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} - 1 \right) \end{aligned} \right\} \leq - \frac{1}{D_n} - \frac{1}{D_n \cdot l_1 D_n} - \dots - \frac{1}{D_n \cdot l_1 D_n \dots l_{p-1} D_n} \\ - \frac{\mu}{D_n \cdot l_1 D_n \dots l_p D_n} - O\left(\frac{1}{D_n^\lambda}\right) \quad (\lambda > 1),$$

for  $n \geq m$ , where  $m$  is so large that  $l_p D_m > 0$ , then the condition  $\mu > 1$  is sufficient for the convergence of  $\Sigma a_n$ .

Theorem VI suggests that this result may be supplemented by the following.

If, in the last relation, the equality sign prevails, then either of the conditions:

$$(i) \mu > 1, \quad (ii) \lim_{n \rightarrow \infty} \frac{a_n \cdot D_n \cdot l_1 D_n \dots l_p D_n}{d_n} = 0$$

is necessary and sufficient for the convergence of  $\Sigma a_n$ .

The proof of this is similar to the proof of Theorem VI and depends on the two corollaries under Theorem I.

**THEOREM VII.** If  $\Sigma (-1)^{n-1} a_n$  ( $a_n > 0$ ) is an alternating series,  $(D_n)$  is a sequence as in Theorem A and

$$\left. \begin{aligned} & \frac{1}{d_n} \log \frac{a_{n+1}}{a_n} \\ \text{or, } & \frac{1}{d_n} \left( \frac{a_{n+1}}{a_n} - 1 \right) \end{aligned} \right\} = - \frac{\mu}{D_n} - O\left(\frac{1}{D_n^\lambda}\right) \quad (\lambda > 1),$$

then either of the conditions: (i)  $\mu > 0$ , (ii)  $\lim_{n \rightarrow \infty} a_n = 0$ , is necessary and sufficient for the convergence of  $\Sigma (-1)^{n-1} a_n$ . C. T. R.

\* It may be noticed that if  $(D_n)$  is any strictly increasing divergent sequence,  $\left(\frac{a_n}{d_n}\right)$  a positive decreasing sequence, the condition  $\lim_{n \rightarrow \infty} \frac{a_n D_n}{d_n} = 0$  is necessary for the convergence of  $\Sigma a_n$ . From this result we can deduce the divergence of  $\Sigma \frac{d_n}{D_n}$  by taking  $a_n = \frac{d_n}{D_n}$ .

† On an integral test of R. W. Brink for the convergence of series, *Bull. Amer. Math. Soc.* 43 (1937), 405-412, Th. 3.

## CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

## REFERENCE FOR SIMILAR TRIANGLES.

SIR,—The letter by Mr. Wood suggests that the symbol  $\sim$  for Similarity is fairly widely used. As he stated, there is not much likelihood of confusion between the two uses of the symbol  $\sim$ , and its adoption in England would tend to uniformity in at least the English-speaking countries of the world. For that reason, I am prepared to withdraw my suggestion in favour of that by Mr. Wood.

It would serve a useful purpose if the Teaching Committee considered the suggestions which have so far been put forward, and others which may come along, and select the one which they think is the best. This one should then be recommended as the standard one to be adopted for England.

Yours truly,

S. INMAN.

## OBTUSE ANGLING—A CATCH.

DEAR SIR,—In discussing the probability that a triangle is obtuse-angled, Mr. Tuckey (Note 1408) is fishing in deep waters. May I offer him a red herring, landed with a hook baited for me by Mr. Robson?

Since a triangle cannot have more than one obtuse angle, the chance that  $ABC$  has an obtuse angle is not greater (one is tempted to say, is less, but wait!) than the sum of the chances that the individual angles are obtuse, that is, than three times the chance that  $C$  is obtuse. But if  $A, B$  are given,  $C$  is obtuse only if  $C$  is inside the circle on  $AB$  as diameter, and the chance of this is zero, since  $C$  can be anywhere in the plane.

If  $H$  is the orthocentre, the chance that  $HBC$  is obtuse-angled is zero, and therefore the chance that  $ABC$  is acute-angled is zero.

There aren't any triangles.

Yours untruly,

E. H. NEVILLE.

## BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, i.e. volume, page, and number. The names of those sending the questions will not be published.

The Secretary would be glad to receive any solutions that have not yet been returned.

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1299. Motorist: I have had a lot of motoring experience, so you can take it from me that if a car is travelling at 30 miles an hour, the driver is also travelling at 30 or thereabouts.—*Daily Herald*, March 23, 1939. [Per Mr. T. R. Dawson.]

MATHEMATICAL NOTES.

1419. *Notes on Conics. 1: The Orthoptic Locus.*

1. Let  $O$  be the intersection of two perpendicular tangents to a conic, let  $SY$ ,  $SZ$  be the perpendiculars from a focus  $S$  to these tangents, and let  $X$  be the foot of the directrix corresponding to  $S$ . Then

$$\begin{aligned} OS^2 &= SY^2 + SZ^2 \\ &= e^2(XY^2 + XZ^2) \\ &= e^2(XO^2 + XS^2) \\ &= \frac{1}{2}e^2(OS^2 + OR^2), \end{aligned}$$

where  $R$  is the reflection of  $S$  in  $X$ . Hence

$$(2 - e^2)OS^2 = e^2 \cdot OR^2,$$

and  $O$  is on an Apollonian locus if  $e < \sqrt{2}$ .

The use of the rectangle  $SYOZ$  is the same in this proof as in the proof which appeals to the equality  $CY^2 + CZ^2 = CO^2 + CS^2$ , but the parabola does not require separate treatment. On the other hand, the extension to confocal conics is not obvious, for two terms  $e_1 \cdot X_1Y^2$ ,  $e_2 \cdot X_2Z^2$  will not combine unconditionally.

2. Let us modify slightly the relation by which the locus of  $Y$  and  $Z$  is recognised. If the radical axis of a coaxial family which has  $S$  for a limiting point is  $u$ , and if  $P_u$  denotes the distance of a point  $P$  from  $u$ , measured with a definite sign, the ratio of  $SP^2$  to  $P_u$  is constant along a circle of the family. To avoid specifying a positive direction we may say that the ratio of  $SP^2$  to  $S_uP_u$  is a numerical constant. Thus the locus  $|SY| = e|XY|$  can be expressed alternatively as  $h \cdot SY^2 = k \cdot S_uY_u$ , where  $u$  is the perpendicular bisector of  $SX$  and  $h:k$  is determinable in terms of  $e$ . For the point  $A$  which is the point between  $S$  and  $X$  satisfying the condition  $SA = e \cdot AX$  we have

$$SA = \{2e/(1+e)\}S_u, \quad A_u = \{(1-e)/(1+e)\}S_u;$$

hence  $4he^2 = k(1-e^2)$ , and the explicit equation of the auxiliary circle is  $(e^{-2} - 1)SP^2 = 4S_uP_u$ , an equation which includes the case of the parabola since it reduces to  $P_u = 0$  when  $e = 1$ . We have now

$$\begin{aligned} (e^{-2} - 1)SO^2 &= (e^{-2} - 1)(SY^2 + SZ^2) \\ &= 4S_u(Y_u + Z_u) \\ &= 4S_u(O_u + S_u); \end{aligned}$$

the sum  $O_u + S_u$  is expressible as  $O_v$ , where  $v$  is the line parallel to  $u$  at distance  $-S_u$  away, that is to say, is the directrix itself, and since  $S_v = 2S_u$ , the equation of the orthoptic locus is

$$(e^{-2} - 1)SO^2 = 2S_vO_v.$$

Writing this equation as

$$\{(2e^{-2} - 1) - 1\}SO^2 = 4S_vO_v$$

we see that the orthoptic locus exists if  $2e^{-2} - 1$  is positive, that is, if  $e < \sqrt{2}$ .

To combine two equations

$$(e_1^{-2} - 1)SY^2 = 4S_{u_1}Y_{u_1}, \quad (e_2^{-2} - 1)SZ^2 = 4S_{u_2}Z_{u_2}$$

it is not necessary that the corresponding constant factors should be identical but only that the two sets of coefficients should be proportional. If  $SX_1/(e_1^{-2} - 1) = SX_2/(e_2^{-2} - 1)$ , which is the condition for the two conics to be confocal, we can write the second equation as  $(e_1^{-2} - 1)SZ^2 = 4S_{u_1}Z_{u_2}$ , and we have

$$\begin{aligned} (e_1^{-2} - 1)SO^2 &= 4S_{u_1}(Y_{u_1} + Z_{u_2}) \\ &= 4S_{u_1}(O_{u_1} + S_{u_2}) \\ &= 4S_{u_1}O_v \end{aligned}$$

where  $v$  is the line parallel to the directrices which cuts the focal axis in the point  $R$  such that  $S$  and  $R$  are isotomic for  $X_1$  and  $X_2$ . Since  $S_v = SX_1 + SX_2$ ,  $SX_1/(e_1^{-2} - 1) = S_v/(e_1^{-2} + e_2^{-2} - 2)$ , and the orthoptic equation is  $(e_1^{-2} + e_2^{-2} - 2)SO^2 = 4S_vO_v$ . E. H. N.

#### 1420. *A Misquoted Title.*

In a Note with this heading (*Gazette*, v. 12, p. 109) I protested against the reading *Essai pour les Coniques* for the title of Pascal's famous paper, and referred to three sources that seemed authoritative for the reading *Essais*. But from photographs of the original sheet I find that my authorities were at fault. The title is in the clearest possible type, and the word is *Essay*. I can only suppose that at some stage of the preparation of Bossut's edition, 1779, a  $y$  in manuscript was misread *is*—with a long-tailed  $s$  this would not be difficult—and that later editors either accepted Bossut's version without investigation or made the correction that would be the obvious one to anyone unfamiliar with *Essay* as a French word.

In his Pascal bibliography, Mairé transcribes correctly in treating the original sheet as an independent item, and also in analysing the definitive edition of Brunschvicg and Boutroux, but he replaces *Essais* by *Essai* in analysing the earlier editions of Bossut, Berthelot, and Lahure. Archibald's transcription, *Scripta Mathematica*, v. 4, p. 323 (1936), is of course accurate. E. H. N.

#### 1421. *Rational triangles.*

Note 1338 refers to the determination of triangles with rational sides and area. Below is given a method of obtaining such triangles.

With the usual notation we have

$$\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C = \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C, \dots\dots\dots(i)$$

and

$$\begin{aligned}\sin A &= (2 \tan \tfrac{1}{2}A)/(1 + \tan^2 \tfrac{1}{2}A) \\ &= (2 \cot \tfrac{1}{2}A)/(1 + \cot^2 \tfrac{1}{2}A). \dots\dots\dots(ii)\end{aligned}$$

If  $\cot \tfrac{1}{2}A = x$ ,  $\cot \tfrac{1}{2}B = y$ , then (i) gives

$$\cot \tfrac{1}{2}C = (x + y)/(xy - 1).$$

Then substitution in (ii) and similar formulae gives

$$\left. \begin{aligned}\sin A &= 2x/(1 + x^2) \\ \sin B &= 2y/(1 + y^2) \\ \sin C &= 2(x + y)(xy - 1)/(1 + x^2)(1 + y^2)\end{aligned} \right\} \dots\dots\dots(iii)$$

Now let  $4R = (1 + x^2)(1 + y^2). \dots\dots\dots(iv)$

Using (iii), (iv) and the sine rule,

$$a = x(1 + y^2), \quad b = y(1 + x^2), \quad c = (x + y)(xy - 1). \dots\dots\dots(v)$$

If  $x, y$  be rational and positive and  $xy > 1$ , (v) gives a set of rational and positive sides. Also

$$\begin{aligned}\Delta &= \tfrac{1}{2}bc \sin A \\ &= xy(x + y)(xy - 1),\end{aligned}$$

and so the area also is rational.

Multiplication by a suitable factor will make the results integral.

J. H. CADWELL.

#### 1422. On gauge constructions.

§ 1. We take as our basic constructions, not the usual ruler and compass constructions, but the following :

1. Given two points, the line joining them can be constructed ;
2. If two lines meet, their common point can be constructed ;

and one of the following :

- a An interval congruent to any given interval can be cut off from any given ray ;
- or b We are given, once for all, a fixed interval called the "gauge". An interval congruent to the gauge can be cut off from any given ray (we denote the gauge by  $u$ ) ;
- or c We are given, once for all, a fixed interval, called the "gauge", and a fixed point  $O$ . An interval congruent to the gauge can be cut off from any ray from  $O$ , one end of the interval being at  $O$ .

Construction  $c$  is weaker than  $b$ , and  $b$  is weaker than  $a$ .

Hilbert (*Grundlagen der Geometrie*, 1930 and earlier editions, chapter 7) discusses construction  $b$  in the presence of the parallel axiom. Forder (*Euclidean Geometry*, 1927) shows that if we assume the parallel axiom, then from construction  $c$  we get the same results as from  $b$  (*l.c.* p. 217 ff.). He also discusses construction  $a$  without the parallel axiom (*l.c.* p. 107 ff.), but there the argument is in the service of the axiomatic treatment of congruence ; the question of construction is quite secondary.

It is very unlikely that  $c$  can be utilised without the parallel axiom. The purpose of this note is to show that if we assume  $b$  we can essentially do all constructions which can be performed by  $a$ , without using the parallel axiom.

Thus Hilbert's results can be improved in two directions: (i) it is not necessary to use the parallel axiom if we assume  $b$ ; (ii) if we assume the parallel axiom, we can replace  $b$  by  $c$ .

§ 2. We assume constructions 1, 2,  $b$ . Our main object is to solve  $a$ , but in doing so, we show that most of the congruence constructions can be performed. The figures, being simple, are not drawn.

§ 3. *To bisect a given angle  $AOB$ .* Cut off the gauge  $u$  on  $OA$  and  $OB$  twice, so that  $OX = XY = u$  on  $OA$ , and  $OX' = X'Y' = u$  on  $OB$ . Join  $X'Y$ ,  $XY'$ ; the cut of  $X'Y$ ,  $XY'$  is on the required bisector. Join this cut to  $O$ .

§ 4. *To draw some perpendicular to a given line  $OA$ .* Draw any ray  $OB$  from  $O$ . Bisect angle  $AOB$  by  $OX$  [3]. Take  $P$ ,  $Q$ ,  $R$  on  $OA$ ,  $OB$ ,  $OX$  so that  $OP = OQ = OR = u$  (the gauge). Bisect angle  $AOX$  by  $OY$  [3]. Let  $OY$  cut  $PQ$  in  $S$ . Then  $PQ \perp OX$ ; hence  $RS \perp OA$ .

§ 5. *To draw a perpendicular to a given line  $OA$  at a given point  $A$ .* By [4] we can draw a perpendicular  $PBQ$  to  $OA$ . Let  $B$  be its foot, and  $BP = BQ = u$  (the gauge). If  $B$  is not at  $A$ , let  $Q'$  be any point on the ray  $QA$  beyond  $A$ . Bisect angle  $PAQ'$  [3]. This line is the required perpendicular.

§ 6. *To bisect a given interval  $AB$ .* Draw  $PA$ ,  $BQ$  perpendicular to  $AB$  [5]. Let  $P$ ,  $Q$  be on opposite sides of  $AB$  and  $PA = QB = u$ . The join  $PQ$  cuts  $AB$  in its midpoint.

§ 7. *If  $OX$ ,  $OY$  be any rays from  $O$ , to find  $Z$  on  $OY$  so that  $OX = OZ$ .* Bisect angle  $XOY$  by  $OP$  [3]. Take  $A$  on  $OX$ ,  $B$  on  $OY$  so that  $OA = OB = u$ . Let  $XB$  cut  $OP$  in  $R$ ; then  $AR$  cuts  $OY$  in the required point  $Z$ .

§ 8. *To find a point  $B$  on any line  $OA$  so that  $OA = OB$ .* Through  $O$  draw any ray  $OP$ , and on it take  $C$  so that  $OA = OC$  [7]. On the line  $OA$ , on the opposite side of  $O$  from  $A$ , take  $B$  so that  $OB = OC$  [7].

§ 9. *If  $O$ ,  $A$ ,  $B$  be collinear points, to find  $C$  on  $OA$  so that  $BC = OA$ .* Bisect the interval  $AB$  at  $P$  [6]. On  $OA$  take  $C$  so that  $PC = PO$  [8]. Then  $C$  is the required point.

§ 10. *On a given ray  $OX$  to cut off from  $O$  an interval congruent to any given interval  $AB$ .* On  $OA$  cut off from  $A$  an interval  $AY = AB$  [7]. On  $OA$  cut off from  $O$  an interval  $OZ = AY$  [9]. On  $OX$  cut off from  $O$  an interval congruent to  $OZ$  [7].

Thus finally we have deduced construction  $a$  from  $b$ , without using the parallel axiom.

There is now no difficulty in performing the following constructions without using the parallel axiom:

§ 11. To draw a perpendicular to a given line from a given point outside it.

§ 12. To construct an angle with a given vertex and arm, congruent to a given angle.

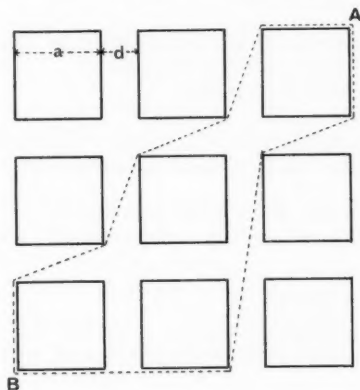
§ 13. On a given base to construct a triangle congruent to a given triangle.

If we now adjoin the parallel axiom, we can construct a parallel to a given line through a given point *via* perpendiculars; but, as we have said, if this axiom is used, we can reduce *b* to *c*.

H. G. FORDER.

#### 1423. A problem on paths.

The problem solved by Mr. Goodstein in Note 1364 of the May Gazette suggests a more complex geodesic difficulty which might be of



some interest to other readers. The problem might be expressed:

"The boundary of Squaretown is a square whose sides, running north-south or east-west, are of length  $na + (n-1)d$ . The town is divided into  $n^2$  blocks of side  $a$  by  $(n-1)$  streets from north to south and  $(n-1)$  streets from east to west, each street being of width  $d$ . There is also a street—of unspecified width—encircling the whole town. What is the shortest distance from the extreme north-east corner to the extreme south-west corner of the town, and how many possible routes are of this minimum length?" Two possible routes are shown.

D. J. FINNEY.

#### 1424. Division without figures.

Arithmetical teasers in which is given the "shape" or "skeleton" of a problem in addition, subtraction, multiplication or division, omitting all but a few of the actual figures involved, are familiar. Recently I was shown such a division sum, in which was given only

one of the thirty figures involved. The last step towards "perfection" can be taken by the use of recurring decimals in the quotient, as shown below.

*Problem.* To complete the division sums.

$$(1) \text{xx}) \text{xx} \quad (\cdot \dot{\text{xxxxx}}$$

$$\begin{array}{r} \text{xxx} \\ \text{xx} \\ \text{xxx} \\ \hline \text{x} \\ \text{xxx} \\ \text{xx} \\ \text{xxx} \\ \hline \text{xx} \end{array}$$

$$(2) \text{xx}) \text{xx} \quad (\cdot \dot{\text{xxxxxx}}$$

$$\begin{array}{r} \text{xxx} \\ \text{xx} \\ \text{xxx} \\ \hline \text{x} \\ \text{xx} \\ \text{x} \\ \text{xxx} \\ \hline \text{xx} \end{array}$$

*Solutions :*

(1) 5 recurring figures in the quotient, so the divisor is a factor of 99,999, and, if prime, is of the form  $5n + 1$ .

The divisor is easily found—41.

The rest follows.

(2) 6 recurring figures in the quotient. Prime numbers giving 6 are of form  $6n + 1$ ; 7 and 13 suit. Hence the divisor is 21, 63, 77, 13, 39 or 91.

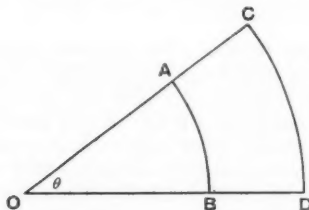
Examination of the skeleton indicates 91.

(11  $\times$  27 and 11  $\times$  37 will give 6 places, but are numbers of 3 digits.)

A. G. SILLITTO.

#### 1425. *A note on elementary calculus.*

This note gives a simple illustration of the fact that if two variables decrease without limit, their limiting ratio may have any value.



In the figure  $AB$  and  $CD$  are arcs of any two concentric circles subtending the same angle  $\theta$  at the centre. Since arcs subtending the same angle at the centre of two concentric circles are proportional to their radii,

$$\text{arc } AB : \text{arc } CD = OA : OC.$$

Let  $\theta \rightarrow 0$  so that  $AB \rightarrow 0$ ,  $CD \rightarrow 0$ . Then the limiting ratio of  $AB/CD$  is equal to the ratio of the radii  $OA/OC$  and may have

any value. If, for example,  $OA = \frac{1}{2}OC$ ,

$$\lim_{\theta \rightarrow 0} (AB/CD) = \frac{1}{2};$$

this means that the arc  $AB$  approaches zero half as fast as the arc  $CD$  does.

F. M. MARZIALS.

1426. *On Note 1395 (XXIII, p. 301).*

Dr. F. G. Maunsell asks for particulars of the problem to find three integers  $x, y, z$  whose squares are in arithmetical progression. This problem is fully discussed in chapter xiv of L. E. Dickson's *History of the Theory of Numbers*, vol. 2. A particular solution was given by Diophantus about A.D. 300. It was treated more generally by Jordanus Nemorarius in 1496 and again by Vieta in 1591 who solved for  $x, y, z$  in the form

$$a^2 - 2b^2, a^2 + 2b^2 + 2ab, a^2 + 2b^2 + 4ab.$$

In 1636 Fermat proposed to St. Croix the same problem only modified by having a perfect square for the common difference in the progression. Later again Fermat gave a solution of the original problem which is closely allied to that given by Frénicle in his great work on rational right-angled triangles in 1676, namely

$$p^2 - 2pq - q^2, p^2 + q^2, p^2 + 2pq - q^2,$$

which is essentially the solution given by Maunsell and is easily transformable into that of Vieta.

The problem was also familiar to James Gregory, who used it, sometime in the period 1672-1675, for his proof by descent of the rather difficult problem of Fermat on the area of a rational right-angled triangle—to prove that such an area (and also twice such an area) could not be equal to the square of a rational. This occurs in the hitherto unpublished manuscript notes of Gregory at St. Andrews.

Many later writers have also considered the problem.

H. W. TURNBULL.

1427. *Vandermonde's Theorem and some associated formulae.*

In this Note the following notation is used :

$$n^{(r)} \equiv n(n-1)(n-2) \dots (n-r+1),$$

$$n_{(r)} \equiv n^{(r)}/r! \equiv \binom{n}{r},$$

where  $n$  is unrestricted but  $r$  is a positive integer; by extension and analogy  $n_{(0)} = 1$ . For any value of  $r$ ,  $0_{(r)} = 0$ . When  $n$  is negative we may use the form

$$(1) \quad (-n)_{(r)} = (-)^r (n+r-1)_{(r)}.$$

In this notation Vandermonde's Theorem as generally given is

$$(2) \quad (a+b)^{(n)} = a^{(n)} + na^{(n-1)}b + n_{(2)}a^{(n-2)}b^{(2)} + \dots + n_{(r)}a^{(n-r)}b^{(r)} + \dots + b^{(n)}.$$

A simpler and more convenient form, involving only binomial coefficients, is found by dividing both sides by  $n!$ , giving

$$(3) \quad (a+b)_{(n)} = a_{(n)} + a_{(n-1)}b + a_{(n-2)}b_{(2)} + \dots + a_{(n-r)}b_{(r)} + \dots + b_{(n)}.$$

Or, putting  $-b$  for  $b$  and using (1),

$$(4) \quad (a-b)_{(n)} = a_{(n)} - a_{(n-1)}b + a_{(n-2)}(b+1)_{(2)} - \dots \\ + (-)^r a_{(n-r)}(b+r-1)_{(r)} \dots + (-)^n (b+n-1)_{(n)}.$$

Form (3), which may be regarded as standard, results immediately (whatever the values of  $a$  and  $b$ ) from Newton's Advancing Difference Interpolation Formula,

$$(5) \quad u_{a+x} = u_a + x\Delta u_a + x_{(2)}\Delta^2 u_a \dots,$$

in which we put  $x=b$ ,  $u_a \equiv a_{(n)}$ , and therefore  $\Delta^t u_a = a_{(n-t)}$  as is easily shown.

An alternative form comes in precisely the same way from Newton's Backward Difference Interpolation formula, viz.:

$$(6) \quad u_{a+x} = u_a + x\Delta u_{a-1} + (x+1)_{(2)}\Delta^2 u_{a-2} + (x+2)_{(3)}\Delta^3 u_{a-3} + \dots,$$

which gives the variant formula

$$(7) \quad (a+b)_{(n)} = a_{(n)} + (a-1)_{(n-1)} \cdot b + (a-2)_{(n-2)} \cdot (b+1)_{(2)} \\ + (a-3)_{(n-3)} \cdot (b+2)_{(3)} + \dots (b+n-1)_{(n)};$$

or, putting  $-b$  for  $b$ ,

$$(8) \quad (a-b)_{(n)} = a_{(n)} - (a-1)_{(n-1)} \cdot b + (a-2)_{(n-2)} \cdot b_{(2)} \\ - (a-3)_{(n-3)} \cdot b_{(3)} \dots \pm b_{(n)}.$$

Each alternative interpolation formula, such as Bessel's, Stirling's and Everett's, will give a variant form of the identity, which can be readily written down, always remembering that  $\Delta^t a_{(n)} = a_{(n-t)}$ . Such identities may sometimes be useful, but it is unnecessary to record them since they can be so easily reproduced.

A number of simple identities may be found by making  $(a \pm b)$  equal to  $n$ , or alternatively to any of  $0, 1 \dots (n-1)$ .

I. If  $(a \pm b) = n$ , the L.H.S. of (3) or (4) takes the value  $n_{(n)} = 1$ . Thus for any value of  $a$  and  $n$ , (3) gives

$$(9) \quad 1 = a_{(n)} + a_{(n-1)} \cdot (n-a) + a_{(n-2)} \cdot (n-a)_{(2)} + \dots + (n-a)_{(n)}$$

and (4) gives

$$(10) \quad 1 = a_{(n)} - a_{(n-1)} \cdot (a-n) + a_{(n-2)} \cdot (a-n+1)_{(2)} - \dots \pm (a-1)_{(n)}.$$

II. If  $(a \pm b)$  has any value, say  $t$ , ranging from  $0$  to  $n-1$  inclusive, the L.H.S. of (3) or (4) vanishes. Hence if  $t$  is so restricted, (3) gives for any value of  $a$ :

$$(11) \quad 0 = a_{(n)} + a_{(n-1)}(t-a) + a_{(n-2)} \cdot (t-a)_{(2)} + \dots + (t-a)_{(n)};$$

and (4) gives for any value of  $a$ :

$$(12) \quad 0 = a_{(n)} - a_{(n-1)} \cdot (a-t) + a_{(n-2)} \cdot (a-t+1)_{(2)} - \dots \pm (a-t+n-1)_{(n)}.$$

G. J. LIDSTONE.

1428. *Another proof of the identity in Note 1310.*

Let  $t_r \equiv \phi^r + \psi^r$ ,

and let  $\phi + \psi = \phi\psi = p$ , say.

Then  $\phi, \psi$  are roots of the equation  $x^2 - px + p = 0$ ,  
and from this is easily obtained

$$t_r = p(t_{r-1} - t_{r-2}) \dots\dots\dots(1)$$

Let  $s_n$  denote the sum of

$$t_n + \begin{bmatrix} n \\ 1 \end{bmatrix} t_{n-1} + \begin{bmatrix} n \\ 2 \end{bmatrix} t_{n-2} + \dots + \begin{bmatrix} n \\ n-2 \end{bmatrix} t_2 + \begin{bmatrix} n \\ n-1 \end{bmatrix} t_1.$$

Then, using (1) for all terms save the last,

$$\begin{aligned} s_n &= p(t_{n-1} - t_{n-2}) + p \cdot \begin{bmatrix} n \\ 1 \end{bmatrix} (t_{n-2} - t_{n-3}) + \dots \\ &\quad + p \begin{bmatrix} n \\ r \end{bmatrix} (t_{n-r-1} - t_{n-r-2}) + \dots + p \begin{bmatrix} n \\ n-2 \end{bmatrix} (t_1 - 2) + \begin{bmatrix} n \\ n-1 \end{bmatrix} t_1 \\ &= p \left\{ t_{n-1} + \begin{bmatrix} n-1 \\ 1 \end{bmatrix} t_{n-2} + \dots + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} t_{n-r} + \dots + \begin{bmatrix} n-1 \\ n-2 \end{bmatrix} t_1 \right\} \\ &\quad - 2p \begin{bmatrix} n \\ n-2 \end{bmatrix} + \begin{bmatrix} n \\ n-1 \end{bmatrix} t_1, \end{aligned}$$

using the easily verified relationship

$$\begin{bmatrix} n \\ r-1 \end{bmatrix} - \begin{bmatrix} n \\ r-2 \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}.$$

Then, noting that  $p = t_1$ , and that

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = 2 \begin{bmatrix} n \\ n-2 \end{bmatrix},$$

we obtain

$$s_n = p s_{n-1}.$$

But  $s_1 = p$ , and so  $s_n = p^n$ .

A. J. HULL.

1429. *An algebraic note.*

If  $y = (ax^2 + bx + c)/(a'x^2 + b'x + c')$ , .....(i)

most elementary textbooks on algebra show how to find the restrictions on the values of  $y$  when  $x$  is real. The method, of course, is to clear the equation of fractions and solve the resulting quadratic equation in  $x$ . This gives

$$x = [b - b'y \pm \sqrt{\{(b - b'y)^2 - 4(a - a'y)(b - b'y)\}}]/(a - a'y). \dots\dots(ii)$$

Thus the quadratic expression in  $y$  under the square root sign, which may be written  $Ay^2 + 2By + C$ , cannot be negative. If the roots of  $Ay^2 + 2By + C = 0$  are imaginary, then for a real  $x$ , if  $A$  is positive  $y$  will have no restriction, but if  $A$  is negative  $y$  will have no possible value.

If the roots of  $Ay^2 + 2By + C = 0$  are real, say  $\alpha$  and  $\beta$ , then

$$Ay^2 + 2By + C \equiv A(y - \alpha)(y - \beta). \dots\dots\dots(iii)$$

In this case if  $A$  is positive  $y$  cannot lie within the interval from  $\alpha$  to  $\beta$ , but if  $A$  is negative  $y$  cannot lie outside that interval.

The object of this note is to show how to set an example of these latter kinds with given values of  $\alpha$  and  $\beta$ .

It might be supposed that it would be sufficient to work backwards, and write

$$x = [ly + m \pm \sqrt{\{A(y - \alpha)(y - \beta)\}}] / n(y - c). \dots\dots\dots(iv)$$

But this, in general, gives when rationalised a *quadratic* equation in  $y$ , namely

$$\{xn(y - c) - (ly + m)\}^2 = A(y - \alpha)(y - \beta). \dots\dots\dots(v)$$

In fact, since (iv) has 7 disposable constants, while (i) has but 6, there must be a relation between the 7 constants if (iv) is to be equivalent to (i). Writing (v) in the form

$$\{xn(y - c)\}^2 - 2xn(y - c)(ly + m) = A(y - \alpha)(y - \beta) - (ly + m)^2,$$

we see that the condition that the factor  $y - c$  should be common to both sides is

$$A(c - \alpha)(c - \beta) = (lc + m)^2. \dots\dots\dots(vi)$$

This is the required condition to be satisfied by the constants of (iv). We can then remove the factor  $y - c$  and get an equation of the first degree in  $y$ . Solving this, we get an expression like (i).

Supposing we want an example in which  $y$  is not to lie outside the interval from  $\alpha$  to  $\beta$ , the  $A$  in (iv) must be negative, and hence (vi) shows that  $(c - \alpha)(c - \beta)$  must also be negative; that is, we must choose  $c$  to lie between  $\alpha$  and  $\beta$ .

As an example, take  $\alpha = 2$ ,  $\beta = 5$ ,  $c = 3$ . Then (vi) becomes

$$A \cdot 1 \cdot (-2) = (3l + m)^2.$$

To avoid irrationals choose  $-2A$  to be an exact square, say 4. Then  $A = -2$ . The simplest values of  $l$  and  $m$  will then be  $l = 1$ ,  $m = -1$ . Taking  $n = 1$ , (iv) reduces to

$$x = [y - 1 \pm \sqrt{\{-2(y - 2)(y - 5)\}}] / (y - 3).$$

Rationalising, we get

$$\{x(y - 3) - y + 1\}^2 + 2(y^2 - 7y + 10) = 0,$$

whence  $(y - 3)\{x^2(y - 3) - 2x(y - 1) + 3y - 7\} = 0$ ,

and dropping the factor  $y - 3$  and solving for  $y$ ,

$$y = (3x^2 - 2x + 7) / (x^2 - 2x + 3),$$

which is the example required.

If we want an example in which  $y$  is not to lie inside the interval from  $\alpha$  to  $\beta$ , then  $c$  must be chosen to lie outside that interval, so that by (vi)  $A$  will be positive.

For an example of this type, take the same values  $\alpha=2$ ,  $\beta=5$ , but  $c=0$ . Then (vi) becomes  $2 \cdot 5 \cdot A = m^2$ . Taking  $m=10$ , we have  $A=10$ , and (iv) becomes

$$x = [ly + 10 \pm \sqrt{\{10(y-2)(y-5)\}}]/ny.$$

Here, on rationalising and dropping the factor  $y$  we get

$$y(xn-l)^2 - 20(xn-l) = 10y - 70$$

or

$$y = \{20(xn-l) - 70\} / \{(xn-l)^2 - 10\}.$$

Thus, whatever values we may choose for  $n$  and  $l$ , the fraction will have the required property.

Another method is to take a fraction having *any* limits, for example, one given by Hall and Knight :

$$y = (x^2 + x + 1)/(x^2 - x + 1).$$

This cannot have any value outside the interval from  $\frac{1}{3}$  to 3.

Now let

$$y = pz + q,$$

and choose  $p$  and  $q$  so that

$$\frac{1}{3} = p\alpha + q,$$

$$3 = p\beta + q,$$

whence, solving,

$$p = 8/3(\beta - \alpha), \quad q = (\beta - 9\alpha)/3(\beta - \alpha).$$

Thus

$$\frac{x^2 + x + 1}{x^2 - x + 1} = \frac{8z + \beta - 9\alpha}{3(\beta - \alpha)},$$

whence  $z = \{(3\alpha + \beta)(x^2 + 1) + (2\beta - 6\alpha)x\}/4(x^2 - x + 1)$ .

This is the required fraction.

As a particular example, take  $\alpha=2$ ,  $\beta=5$ .

Then

$$z = \{11(x^2 + 1) - 2x\}/4(x^2 - x + 1),$$

whence

$$x = [2z - 1 \pm \sqrt{\{12(z-2)(5-z)\}}]/(2z-1),$$

which shows that if  $x$  is real,  $z$  cannot lie outside the interval from 2 to 5.

If we want a fraction of type (i) which cannot take values *inside* a given interval from  $\alpha$  to  $\beta$ , we must start with a fraction having a similar property for any other interval. We may start, for example, with

$$(6x^2 - 4x - 6)/(x^2 - 2x - 3),$$

which cannot take any value within the interval from 2 to 5 but may have any other value.

*Note.*—The purpose of the foregoing article is to enable those who have to set examples of this sort to do so with minimum effort.

R. F. MUIRHEAD.

1430. *A proof of the binomial theorem for a positive integral exponent.*

The proof given here has certain merits peculiar to itself, though

I do not suggest that it should take the place of the natural and fundamental proof based on the study of the product

$$(x + a_1)(x + a_2)(x + a_3) \dots (x + a_n).$$

One of its merits is that it is *complete*, not requiring the assumption of the formula

$$n(n-1) \dots (n-r+1)/1 \cdot 2 \cdot 3 \dots r$$

for the number of  $r$ -combinations of  $n$  things. From the examinee's point of view, the proof has the merit of *brevity*. I think no other proof, even assuming the  $r$ -combination formula or based on the proof that if true for any particular value  $n$  of the exponent it is true for the value  $n+1$ , can be stated *completely* in so few words.

The proof is as follows: we can obviously assume that the expansion of  $(a+b)^n$  is of the form

$$a^n + (n, 1) a^{n-1}b + \dots + (n, r) a^{n-r}b^r + \dots + b^n,$$

where  $(n, 1), \dots (n, r), \dots$  are certain numerical constants to be determined. Then:

$$\{(1+x) + y\}^n = (1+x)^n + (n, 1)(1+x)^{n-1}y + \dots + (n, r-1)(1+x)^{n-r+1}y^{r-1} + \dots \dots \dots (i)$$

$$\{1 + (x+y)\}^n = 1 + (n, 1)(y+x) + \dots + (n, r)(y+x)^r + \dots \dots \dots (ii)$$

Equating coefficients of  $xy^{r-1}$  in (i) and (ii),

$$(n, r-1) \cdot (n-r+1, 1) = (n, r) \cdot (r, 1). \dots \dots \dots (iii)$$

Again, equating coefficients of  $x$  in the expansions of  $(1+x)^n$  and  $(1+x)(1+x)^{n-1}$  we get

$$(n, 1) = (n-1, 1) + 1.$$

$$\text{Similarly, } (n-1, 1) = (n-2, 1) + 1,$$

$$\dots \dots \dots$$

$$(2, 1) = (1, 1) + 1$$

$$= 1 + 1.$$

$$\text{Adding, } (n, 1) = 1 + 1 + \dots \text{ to } n \text{ terms}$$

$$= n.$$

Hence (iii) may be written

$$(n, r-1) \cdot (n-r+1) = (n, r) \cdot r;$$

$$\text{similarly, } (n, r-2) \cdot (n-r+2) = (n, r-1) \cdot (r-1),$$

$$\dots \dots \dots$$

$$(n, 0) \cdot n = (n, 1) \cdot 1.$$

Multiplying, we get

$$(n-r+1) \cdot (n-r+2) \dots n = (n, r) \cdot r(r-1)(r-2) \dots 2 \cdot 1$$

$$\text{or } (n, r) = n(n-1)(n-2) \dots (n-r+1)/1 \cdot 2 \cdot 3 \dots r,$$

which proves the theorem.

If we might assume the result  $(n, 1) = n$ , which is fairly obvious, the number of words required in the preceding proof would be reduced to about half.

R. F. MUIRHEAD.

1431. *A logical point in dynamics.*

In Note 1376 in the *May Gazette* on the problem "Given the maximum acceleration and retardation of a moving point, find the least time in which a stated distance can be described from rest to rest", Mr. R. L. Goodstein states that, if we assume that the least time is given by maximum acceleration followed by maximum retardation, then the problem is quite trivial. He then asserts that "the proof of the assumption is well beyond Intermediate standard". Without having put the matter to a practical test, I venture to think that he has overrated the difficulties of the proof and that the average Intermediate student would find the following work both comprehensible and convincing.

For brevity, I describe the velocity-time curve of any motion from rest to rest in which the acceleration and retardation do not exceed  $f_1$  and  $f_2$  respectively as a *suitable* velocity-time curve, and I give the arguments in the form of two theorems.

Let  $S$  be the distance to be traversed and let  $T$  be the least time of describing it under the conditions stated; then

**THEOREM I.** *The velocity-time curve for this motion is a suitable velocity-time curve for the motion in which the greatest possible distance is traversed in time  $T$ .*

For, if not, let it be possible to traverse a distance  $S' (> S)$  in time  $T$  under the conditions stated. Draw the velocity-time curve for this motion; the area contained between it and the  $t$ -axis is  $S'$ . Now, keeping the end-points of this curve on the  $t$ -axis, reduce its size without altering its shape until its linear dimensions are  $\sqrt{(S/S')}$  times their original values. The resulting curve is a suitable velocity-time curve (since slopes are unaltered by the change in scale); the area between it and the  $t$ -axis is  $S$ , and the difference of the abscissae of its end-points is  $T\sqrt{(S/S')}$ , which is less than  $T$ . We have thus determined a motion in which the distance  $S$  is traversed under the conditions stated in a time less than  $T$ . This contradiction proves theorem I.

Now let  $A$  and  $B$  be the end-points of any suitable velocity-time curve for which  $AB = T$ . Draw the line  $l_1$  of slope  $f_1$  through  $A$  and the line  $l_2$  of slope  $-f_2$  through  $B$ , and let these lines meet at  $C$ . (I omit the actual diagram, which the reader should draw for himself.) Let  $P$  and  $P_1$  be points on the curve and on  $l_1$  respectively with the same abscissa; since the slope of the curve never exceeds the slope of the line, the height  $PP_1$  must increase (more precisely must be non-decreasing) as  $PP_1$  moves to the right; and, since this height is zero when  $P$  and  $P_1$  are at  $A$ , it must be positive (non-negative) when  $PP_1$  is on the right of  $A$ . Hence, on the right of  $A$ , the curve is never above  $l_1$ ; for similar reasons, it is never above  $l_2$  on

the left of  $B$ . Consequently no point of the curve is above the "broken line"  $ACB$ , and so the area between the curve and the  $t$ -axis never exceeds the area of the triangle  $ACB$ , and it is equal to the area of the triangle when and only when the curve coincides with the broken line  $ACB$ . This proves

**THEOREM II.** *If a point moving for a time  $T$  has its initial and final velocities zero and has maximum acceleration  $f_1$  and maximum retardation  $f_2$ , then, if the point is to traverse the greatest distance possible, the velocity-time curve of the motion must be the above-defined broken line  $ACB$ .*

The proof of the assumption follows immediately by combining theorem I with theorem II.

It seems to me that the arguments just given should be comprehensible to any student who understands the significance of a velocity-time curve and who has grasped the dynamical interpretations of its area and slope.

It is, of course, easy for any competent analyst to translate into a high-brow arithmetical form the geometrical arguments which have been used to prove theorem II. When this is done, the result obtained may be stated as follows :

For given positive  $f_1, f_2$  and  $S$ , let

$$T = \sqrt{\frac{2S(f_1 + f_2)}{f_1 f_2}}.$$

Let  $f(t)$  be defined "almost always" (i.e. except at a set of instants of measure zero) in a range of values of the time  $t$  whose length is equal to  $T$ . Let  $f(t)$  be integrable throughout the range and, at all instants at which  $f(t)$  is defined, let  $-f_2 \leq f(t) \leq f_1$ . Then it is not possible for a point moving from rest to rest with acceleration  $f(t)$  to travel a distance  $S$  in a sub-interval of  $T$ ; and the point can travel this distance in the whole interval only if  $f(t)$  is first almost always equal to  $f_1$  and then almost always equal to  $-f_2$ .

In conclusion I remark that, although theorem I is proved by a trick, yet this trick is fairly obvious and is easy to remember; moreover, theorem I affords a useful illustration of the general principle that there exists a class of minimal (maximal) theorems which possess related maximal (minimal) theorems. G. N. WATSON.

#### 1432. Interpolation by Proportional Parts of Mean Differences.

Mr. Hope-Jones has pointed out [*ante*, Vol. XXIII, pp. 21-25] that in interpolating between two tabular values, by means of tabulated proportional parts based on mean differences, an increase of accuracy is secured by working forward from the earlier value or backward from the later value according as the interval of interpolation is greater or less than half the tabular interval. The object of this Note is to give an algebraic investigation which will be found broadly to support the rule, while also indicating some considerable theoretical and practical limitations of its generality. It should,

however, be pointed out that when the first differences are changing rapidly enough to make the rule of practical value, much better results will be obtained by ignoring the tabular proportional parts and using parts based on the actual difference between the two tabular values of the function, and this is the course that an experienced computer would adopt. In this case the same result is obtained by forward or backward interpolation.

Let each line of the table contain  $t$  entries [ $t$  is usually 10], and take the origin of  $x$  at the beginning of the line we are dealing with, so that this line contains the values  $u_0, u_1, \dots, u_{t-1}$ ; and suppose we want  $u_{k+\theta}$ ; [ $k$  integral,  $0 \leq k < t$ ;  $0 \leq \theta < 1$ ]. It will be assumed that, as is the case in practice, the error arising from third and higher differences is negligible in comparison with that arising from the second difference, so that the function may be sufficiently represented, over the range of interpolation, by the quadratic expression

$$u_x = A + Bx + Cx^2.$$

Evidently linear interpolation will reproduce  $A$  and the coefficient of  $B$ , and we need only consider the coefficients of  $C$ . The material coefficients of  $C$  are as follow :

Function.	Coefficient of $C$ .
$u_0$	0
$u_t$	$t^2$
$(u_t - u_0)/t = \text{mean diff.}$	$t$
$u_k$	$k^2$
$u_{k+1}$	$k^2 + 2k + 1$
$u_{k+\theta}$	$k^2 + 2k\theta + \theta^2$
$u_{k+\theta}$ by forward interpolation	$k^2 + t\theta$
$u_{k+\theta}$ by backward interpolation	$k^2 + 2k + 1 + t(\theta - 1)$

Thus if  $E$  and  $e$  are the errors in the coefficients of  $C$ , by forward and backward interpolation respectively, we have

$$\begin{aligned} E &= (t - 2k - \theta)\theta, \\ e &= (t - 2k - \theta)\theta - (t - 2k - 1) \\ &= E - (t - 2k - 1), \end{aligned}$$

so that the algebraic difference between  $E$  and  $e$  is independent of  $\theta$ . Also, if  $t - 2k \equiv \lambda$ , we may write

$$\begin{aligned} E &= (\lambda - \theta)\theta, \\ e &= (\lambda - \theta - 1)(\theta - 1). \end{aligned}$$

From this it follows\* that, for the prescribed limits of  $k$  and  $\theta$ ,  $E$  and  $e$  have different signs; for  $\theta$  is positive and  $\theta - 1$  is negative, and it is easily seen that  $\lambda - \theta$  and  $\lambda - \theta - 1$  have the same sign.\*

\* Except when  $\lambda = 1$ , in which case  $E = e = \theta(1 - \theta)$  for all values of  $\theta$ . This case can arise only when  $t$  is odd, as for example  $t = 5$  in a split-line table of 10 values to the line.

The signs will be  $E$  positive,  $e$  negative, or *vice versa* as  $2k < > t - \theta$ :  
i.e. as

$$k < \text{ or } \geq \frac{1}{2}t \text{ (} t \text{ even),}$$

or

$$k < \text{ or } > \frac{1}{2}(t-1) \text{ (} t \text{ odd).}$$

Let  $E'$  and  $e'$  represent the errors at the point  $t - (k + \theta)$ , viz. the point which is symmetrical with  $k + \theta$  round the centre of the line. Then it is easily proved that  $E' = -e$  and  $e' = -E$ . This indeed follows from symmetry, as may be seen by reversing the order of the entries and writing  $v_{t-x} = u_x$ .

Since  $E$  and  $e$  have different signs,  $E$  and  $-e$  will have the same sign, which will be  $+$  if  $\lambda > 0$  and  $-$  if  $\lambda < 0$ . Consider the inequality  $E > -e$ , which reduces on substitution to

$$(A) \quad \theta^2 - \lambda\theta + \frac{1}{2}(\lambda - 1) < 0.$$

It is known [see Hall and Knight's *Algebra*, p. 90, or Chrystal's *Algebra*, Part I, pp. 454-8] that this is true if  $\theta$  lies between the roots of the quadratic, which are

$$\frac{1}{2}\lambda \pm \frac{1}{2}\sqrt{(\lambda^2 - 2\lambda + 2)}.$$

Now if  $\lambda$  is positive,  $E$  and  $-e$  are positive, so that the inequality (A) means that the modulus [or numerical value taken positively] of  $E$  is greater than that of  $e$ , or  $|E| > |e|$ . If  $\lambda$  is negative or zero,  $E$  and  $-e$  are negative, so (A) means that  $|E| < |e|$ . We are concerned only with that part of the range between the roots that falls between 0 and 1; bearing this in mind and calculating the roots, we get the following table of  $L_k$  for  $t=10$  and  $t=5$ , where 0 to  $L_k$  is the range of  $\theta$  over which  $|E| < |e|$ :

	$t=10$									
$k$	0	1	2	3	4	5	6	7	8	9
$\lambda$	10	8	6	4	2	0	-2	-4	-6	-8
$L_k$	.472	.464	.450	.419	.293	.707	.581	.550	.536	.528
	$t=5$									
$k$	0	1	2	3	4					
$\lambda$	5	3	1	-1	-3					
$L_k$	.438	.382	*	.618	.562					

\* For  $k=2$ ,  $E$  and  $e$  are identical for all values of  $\theta$ .

It will be seen that the limiting points  $k + L_k$  may be arranged in pairs equidistant from the central point, as, for example, 4.293 and 5.707 or  $5 \pm .707$ . The values of  $L_k$  differ substantially, in one direction or the other, from the value .5, especially near the middle of the line of values, where however the errors  $E$  and  $e$  are both at their minimum and are very small.

The above theoretical results will be found to be substantially confirmed by trial, though the effects—and particularly the position

of  $L_k$ —may be obscured by accidental combinations of errors arising from the forcing of the last digits of the tabular values.

If, instead of using the tabular proportional parts based on mean differences, we form the parts from the values  $u_k$  and  $u_{k+1}$ , the coefficient of  $C$  in the error is  $\theta(1-\theta)$ , which cannot exceed  $\frac{1}{4}$ . Save in the exceptional case  $\lambda=1$  this coefficient is much less than  $|E|$  and  $|e|$ , especially when  $k+\theta$  is well removed from the centre of the line.

It is unnecessary to take up space with many examples, which can easily be formed by the reader, but a single illustration may be given, viz.  $1.585^{-1}$  from 5-figure tables with arguments at intervals of  $\cdot 01$ . The true value to 5 places is  $\cdot 63091$ .

<i>Forward.</i>	<i>Backward.</i>	<i>By Consecutive Values.</i>
$\cdot 63291$	$\cdot 62893$	$\cdot 63291 - \frac{1}{2}(\cdot 63291 - \cdot 62893)$
$- \quad 209$	$+ \quad 209$	$= \cdot 63291 - \cdot 199$
$\cdot 63082$	$\cdot 63102$	$= \cdot 63092$
$E = -9$	$e = +11$	Error $+1$

Thus with  $k=8$  and  $\theta=\cdot 5$ ,  $|E|$  is less than  $e$ . The alternative method recommended, using adjacent values, is nearly exact.

G. J. LIDSTONE.

### 1433. Gamma Function.

If we write (say)  $x^p/\Gamma(p+1)=x^p/p!=x_p$ , then the usual formulae are much simplified in this notation, e.g. :

$$\frac{\partial}{\partial x} x_p = x_{p-1} = p x_p / x,$$

$$x_p x_{-p} = \sin p\pi / p\pi.$$

Multinomial Theorem :

$$\Sigma x_p y_q z_r = (x+y+z)_n \text{ (summed over } p+q+r=n).$$

Further there appears to be a symmetry between summation over  $p$  and integration over  $x$  :

$$\text{Exponential Theorem,} \quad \sum_{p=0}^{\infty} x_p e^{-x} = 1.$$

$$\text{Gamma-Integral,} \quad \int_0^{\infty} x_p e^{-x} dx = 1.$$

$$\text{Binomial Theorem,} \quad (x+y)_q = \sum_{p=0}^q x_{q-p} y_p.$$

$$\text{Beta-Integral,} \quad y_{p+q} = \int_{x=0}^y (y-x)_p dx_q.$$

Can any reader suggest an explanation?

C. A. B. SMITH.

1434. *The maximum value of the product, and of the sum of partial products, of a number of positive quantities whose sum is given.*

1. Let the  $n$  positive quantities  $a_1, a_2, \dots, a_n$  be such that

$$a_1 + a_2 + \dots + a_n = A \text{ (a constant).}$$

Then, if we replace each of two of these quantities which are unequal by their arithmetic mean, say  $a_1$  and  $a_2$  by  $\frac{1}{2}(a_1 + a_2)$ , the product  $a_1 a_2 \dots a_n$  is increased, and we deduce the well-known fact that  $a_1 a_2 \dots a_n$  is greatest when  $a_1 = a_2 = \dots = a_n$ . Also, the same method shows that  $\Sigma a_1 a_2 \dots a_r$  is greatest ( $r \leq n$ ) when  $a_1 = a_2 = \dots = a_n$ .

[For  $\Sigma a_1 a_2 \dots a_r = P_1 a_1 a_2 + P_2 (a_1 + a_2) + P_3$ , where  $P_1, P_2$  and  $P_3$  are independent of  $a_1$  and  $a_2$ .]

2. The following is a proof, by the use of identities, of the above theorems.

Again, we suppose that

$$a_1 + a_2 + \dots + a_n = A \text{ (a constant),}$$

and we prove by induction that

$$\begin{aligned} A^r = & \frac{A^{r-2}}{(n-1)} \Sigma (a_1 - a_2)^2 + \frac{A^{r-3} \cdot n \cdot 2!}{(n-1)(n-2)} \Sigma a_1 (a_2 - a_3)^2 + \dots \\ & + \frac{A^{r-s} \cdot n^{s-2} (s-1)!}{(n-1)(n-2) \dots (n-s+1)} \Sigma a_1 a_2 \dots a_{s-2} (a_{s-1} - a_s)^2 + \dots \\ & + \frac{A n^{r-3} (r-2)!}{(n-1)(n-2) \dots (n-r+2)} \Sigma a_1 a_2 \dots a_{r-3} (a_{r-2} - a_{r-1})^2 \\ & + \frac{n^{r-2} (r-1)!}{(n-1)(n-2) \dots (n-r+1)} \Sigma a_1 a_2 \dots a_{r-2} (a_{r-1} - a_r)^2 \\ & + \frac{n^{r-1} \cdot r!}{(n-1)(n-2) \dots (n-r+1)} \Sigma a_1 a_2 \dots a_r. \quad (r=2, 3, \dots, n). \quad (I) \end{aligned}$$

This can be easily proved when  $r=2$  or  $3$ ; e.g. putting  $r=2$ , we have

$$A^2 = \frac{\Sigma (a_1 - a_2)^2}{n-1} + \frac{n \cdot 2!}{n-1} \Sigma a_1 a_2,$$

which is true.

Now it can be shown that

$$A \Sigma a_1 a_2 \dots a_r = \Sigma a_1^2 a_2 \dots a_r + (r+1) \Sigma a_1 a_2 \dots a_{r+1}$$

and

$$\Sigma a_1 a_2 \dots a_{r-1} (a_r - a_{r+1})^2 = (n-r) \Sigma a_1^2 a_2 \dots a_r - (r+1) r \Sigma a_1 a_2 \dots a_{r+1},$$

from which we obtain

$$\begin{aligned} (n-r) A \Sigma a_1 a_2 \dots a_r = & \Sigma a_1 a_2 \dots a_{r-1} (a_r - a_{r+1})^2 \\ & + (r+1) n \Sigma a_1 a_2 \dots a_{r+1}. \quad (II) \end{aligned}$$

Let us now assume that (I) is true for some value of  $r$ ; multiplying each side by  $A$  and using (II), we get

$$\begin{aligned}
 A^{r+1} = & \frac{A^{r-1}}{(n-1)} \Sigma (a_1 - a_2)^2 + \frac{A^{r-2} n \cdot 2!}{(n-1)(n-2)} \Sigma a_1 (a_2 - a_3)^2 + \dots \\
 & + \frac{A^{r+1-s} \cdot n^{s-2} \cdot (s-1)!}{(n-1)(n-2) \dots (n-s+1)} \Sigma a_1 a_2 \dots a_{s-2} (a_{s-1} - a_s)^2 + \dots \\
 & + \frac{A^2 \cdot n^{r-3} (r-2)!}{(n-1)(n-2) \dots (n-r+2)} \Sigma a_1 a_2 \dots a_{r-3} (a_{r-2} - a_{r-1})^2 \\
 & + \frac{A \cdot n^{r-2} \cdot (r-1)!}{(n-1)(n-2) \dots (n-r+1)} \Sigma a_1 a_2 \dots a_{r-2} (a_{r-1} - a_r)^2 \\
 & + \frac{n^{r-1} \cdot r!}{(n-1)(n-2) \dots (n-r)} \Sigma a_1 a_2 \dots a_{r-1} (a_r - a_{r+1})^2 \\
 & + \frac{n^r \cdot (r+1)!}{(n-1)(n-2) \dots (n-r)} \Sigma a_1 a_2 \dots a_{r+1},
 \end{aligned}$$

a result which is of the same form as (I) with  $(r+1)$  instead of  $r$ . Hence (I) is proved.

Now the terms on the right-hand side in (I), with the exception of the last, are positive except when  $a_1 = a_2 = \dots = a_n$ , when they are all zero; hence  $\Sigma a_1 a_2 \dots a_r$  is greatest when  $a_1 = a_2 = \dots = a_n$ .

Putting  $r = n$  in (I), we have

$$\begin{aligned}
 A^n = & \frac{A^{n-2}}{n-1} \Sigma (a_1 - a_2)^2 + \frac{A^{n-3} n \cdot 2!}{(n-1)(n-2)} \Sigma a_1 (a_2 - a_3)^2 + \dots \\
 & + \frac{A^{n-s} \cdot n^{s-2} (s-1)!}{(n-1)(n-2) \dots (n-s+1)} \Sigma a_1 a_2 \dots a_{s-2} (a_{s-1} - a_s)^2 + \dots \\
 & + \frac{A \cdot n^{n-3} \cdot (n-2)!}{(n-1)(n-2) \dots 3 \cdot 2} \Sigma a_1 a_2 \dots a_{n-3} (a_{n-2} - a_{n-1})^2 \\
 & + \frac{n^{n-2} \cdot (n-1)!}{(n-1)(n-2) \dots 3 \cdot 2 \cdot 1} \Sigma a_1 a_2 \dots a_{n-2} (a_{n-1} - a_n)^2 \\
 & + \frac{n^{n-1} \cdot n!}{(n-1)(n-2) \dots 3 \cdot 2 \cdot 1} a_1 a_2 \dots a_n, \dots \dots \dots \text{(III)}
 \end{aligned}$$

and it is clear that  $a_1 a_2 \dots a_n$  is greatest when  $a_1 = a_2 = \dots = a_n$ .

It may be noticed too that (I) and (III) give the greatest values of  $\Sigma a_1 a_2 \dots a_r$  and  $a_1 a_2 \dots a_n$  as respectively  ${}_nC_r (A/n)^r$  and  $(A/n)^n$ , results which are obviously correct.

H. J. CURNOW.

**1300.** It remains only to add a study which exemplifies reasoning in its clearest and most precise form. That study is of course mathematics, and of the mathematical studies, chiefly those that use the type of exposition that Euclid employed. In such studies the pure operation of reason is made manifest. The subject matter depends on the universal and necessary processes of human thought. It is not affected by differences in taste, disposition or prejudices. It refutes the common answer of students who, conformably to the temper of the times, wish to accept the principles and deny the conclusions. Correctness in thinking may be more directly and impressively taught through mathematics than in any other way.—Robert M. Hutchins, President of the University of Chicago, in *Harper's Magazine* for Nov. 1936. [Per Mr. E. B. Escott.]

## REVIEWS.

**An Introduction to the Theory of Numbers.** By G. H. HARDY and E. M. WRIGHT. Pp. xvi, 403. 25s. 1938. (Oxford)

Not so many years ago a book with such a title would have suggested an account of number theory limited for the main part to the classical elements and so based chiefly upon the work of Gauss and Dirichlet. The subject has, however, continued to spread in many directions; its ramifications are so extensive, important and full of life that there has been a recent tendency to enlarge considerably the scope of an introductory book. The authors of this book have taken full advantage of this trend and have given a wide interpretation of the title.

The table of contents, as well as the authors' preface, suggests that a more appropriate title might have been "An introduction to some aspects of the theory of numbers". Thus the book contains some account of the following topics:

- (1) The prime numbers.
- (2) The more familiar arithmetical functions, including the partition function; their generating functions, *i.e.* the Dirichlet series or power series associated with them; and their orders of magnitude.
- (3) Diophantine approximation, including Farey series and theorems of Minkowski, Kronecker and Hurwitz.
- (4) Congruences, and of course the law of quadratic reciprocity.
- (5) Irrational numbers, including decimals, continued fractions, transcendental numbers and the transcendence of  $e$  and  $\pi$ .
- (6) Simple arithmetical fields, *e.g.*  $k(\xi)$ , where  $\xi = i, \rho, \sqrt{2}, \sqrt{5}$ ; and quaternions.
- (7) Diophantine equations and also the representation of numbers by sums and differences of squares, cubes and fourth powers.

These subjects are distributed among the twenty-four chapters without great regard to order and to strictly logical development. It would have added greater unity to the book if the matter had been so arranged that chapters and parts of chapters containing connected subjects followed consecutively. The third chapter on primes—namely chapter 22 of the book—might well have been nearer the first two chapters on primes, namely chapters 1 and 2. Similarly, chapter 24 on some more theorems of Minkowski might be nearer chapter 3 containing one of his fundamental theorems. The account of quaternions in the middle of chapter 20 might also have been given in a separate chapter nearer the quadratic fields in chapter 12.

The authors have made their point of view and object quite clear in the preface. It is to produce a book giving an account of material, not too hackneyed, which they consider interesting, entertaining or important, or about which they felt they had something to say. There is, as they state, no question about a systematic development of a general theory, and results are sometimes used long before they are proved, as for example the existence of primitive roots. There is now and then a tendency to give *ad hoc* proofs, which, while striking, are not nearly as significant or far-reaching as the general theory, and are apt to be concerned more with details. There is no suggestion of any really axiomatic development of arithmetic operations. Thus the definition of the addition of two integers is taken for granted, but not that of the division.

There is no doubt, however, that the authors have produced a refreshing and attractive volume, written in a pleasant and readable style. It contains

a wealth of interesting and often unexpected material which has not yet found its way into other so easily accessible books. The authors have spread their nets far and wide in gathering interesting material and their haul is a very nice one indeed. It is really surprising what an immense storehouse they have filled. There is sufficient variety in it to satisfy the most catholic taste and to cater for the reader in all his moods. He may go through the book from cover to cover, or study a chapter here and there, or dip in now and then for a pleasant morsel. In lighter moments he may turn to the theory of the game of Nim, while on more austere occasions he may study the question of Euclidean algorithms in algebraic fields, or the Rogers-Ramanujan identities in the theory of partitions.

The book, which is well printed, nicely set out and easy to read, is sure to extend the circle of those interested in number-theory. It will act as a stimulus to further study and research, since it contains much recent material still occupying the attention of investigators. The notes at the ends of the chapters are particularly valuable, not only from a historical point of view, but also in supplementing the text and in putting the subject matter in its proper perspective.

The book is sure to have a long and successful life. When the inevitable second edition is contemplated, there are, however, several proofs which should be reconsidered. In chapter 10 the authors have not made quite clear what they mean by the numerator and denominator of a convergent to a continued fraction. Thus they really prove that the value of the continued fraction :

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots \frac{1}{a_n}}} \dots\dots\dots(i)$$

is the quotient  $p_n/q_n$ , where the sequences  $p_n$  and  $q_n$  are defined by the usual recurrence formulae :

$$\begin{aligned} p_0 &= a_0, & p_1 &= a_1 a_0 + 1, & p_r &= a_r p_{r-1} + p_{r-2}, & r &\geq 2, \\ q_0 &= 1, & q_1 &= a_1, & q_r &= a_r q_{r-1} + q_{r-2}, & r &\geq 2. \end{aligned}$$

For the sake of clearness let  $P_n, Q_n$  be the numerator and denominator obtained by direct evaluation of the fraction represented by (i), where we may suppose (as is implicitly done) that  $P_n$  and  $Q_n$  are polynomials in the  $a$ 's without a common polynomial factor. Many writers, including the present ones, assert after the induction process alone that they have proved  $p_n = P_n, q_n = Q_n$ . This is not so, as the equality does not follow until it has been proved that  $p_n, q_n$  are relatively prime, which is usually done by proving that

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

Next the account of quaternions given in chapter 20 is based upon a method from an old paper of Dickson's where there were reasons for it irrelevant in the present volume. It has been superseded by Dickson in his later book, *Algebras and their arithmetics*, which contains the better method of approach due to Hurwitz. The points involved are of some interest and it may be worth while to explain the position. They have their origin in the work of Gauss, who was the first to study the arithmetical properties of the complex numbers  $z = x + iy$ , where  $x, y$  are rational numbers. The question of the definition of the complex integers arises immediately. The obvious one is that  $z$  is a complex integer if  $x$  and  $y$  are both rational integers, and this proves to be consistent with the properties of rational integers; e.g. the complex integers form a closed ring in respect of the operations of addition, subtraction

and multiplication defined in the usual way. Further, the obvious definition of a prime number in the new theory proves to be satisfactory.

But the procedure is not so simple with other types of numbers, *e.g.* those of the form  $x + y\sqrt{5}$ , where  $x$  and  $y$  are rational numbers. The obvious definition of an integer is not now satisfactory. The best proves to be that

$$\zeta = a + \frac{1}{2}(1 + \sqrt{5})b$$

is an integer if  $a, b$  are both integers, or in terms of  $x, y$  if either  $x$  and  $y$  are both integers or  $x, y$  are halves of integers and  $x - y$  is an integer. Then with

$$A = -2a - b, \quad B = a^2 + ab - b^2,$$

we have

$$\zeta^2 + A\zeta + B = 0,$$

that is,  $\zeta$  is the root of a polynomial equation with highest coefficient unity and the other coefficients integers. This also serves as a satisfactory definition independent of any particular method of representation. It can be proved that the integers define a closed set in respect of the operations of addition and multiplication (a ring).

The obvious definition of a prime soon brings in other difficulties. Thus suppose we had considered numbers of the form  $x + y\sqrt{-5}$ , where the integers are given by rational integers  $x, y$ . The obvious definition of a prime number breaks down since, as the factorisations

$$21 = 3 \cdot 7 = (4 + \sqrt{-5})(4 - \sqrt{-5}) = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5})$$

suggest, there are three essentially different factorisations of 21 into primes.

Let us now consider quaternions, which are the next generalisation of the number system, and are really the first introduction to hypercomplex number systems where the law of multiplication is more arbitrary than with complex numbers. A quaternion is a number

$$z = a + bi + cj + dk,$$

where  $i, j, k$  play a part similar to  $i$  in complex numbers; thus

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = k, \quad ji = -k.$$

Hence multiplication is no longer commutative and so

$$zz_1 = az_1 + bz_1i + cz_1j + dz_1k$$

is not in general equal to  $z_1z$ . Both of the previous methods of defining integers prove to be unsatisfactory for quaternions. The authors define  $z$  to be a quaternion integer when  $a, b, c, d$  are integers, but point out that some familiar results no longer hold; *e.g.* the existence of a greatest common divisor of two quaternions. Next if we note that  $z$  is the root of the quadratic equation  $z^2 + Az + B = 0$ , where  $A = -2a, B = a^2 + b^2 + c^2 + d^2$ , and call  $z$  an integer when  $A$  and  $B$  are, difficulties also arise, for we may have two integers, for example  $i$  and  $(4i + 3j)/5$ , whose sum is not an integer. The most suitable definition leading to the development of a satisfactory arithmetic is that  $z$  is an integer if either  $a, b, c, d$  are all integers or  $a, b, c, d$  are all halves of odd integers. This is the method usually adopted.

There are always suggestions occurring to any reviewer about what he would like to have seen included in a book, or about how he would have presented parts of it. The following remarks may be considered in this light.

The authors give only proofs not requiring complex function theory, but a chapter involving the use of the complex variable would not have been amiss, since the book is sure to be used by students with some knowledge of this theory. This would have permitted a few obviously interesting remarks about the zeros of the Riemann zeta function either in the text or the

notes. One might dare to wish that they had found it consistent with their purpose to give a proof of the prime number theorem. This might almost be reckoned as introductory, since the proofs are now so short and the theory so much developed. The authors might also have starred (*i.e.* stated without proof) Legendre's theorem on the representation of numbers as sums of three squares; and perhaps have sketched a better result than "every large integer is at most the sum of thirteen cubes". After all, when giving an alternative proof that  $(s-1)\zeta(s) \rightarrow 1$  as  $s \rightarrow 1$ , they use, as they state, a theorem that they have not proved. Further, in dealing with the order of magnitude of the number of divisors of an integer, they take a page in suggesting how a theorem might be proved by the use of the prime number theorem.

While one can appreciate the motives which, as they state, have led the authors to exclude the theory of quadratic forms, the book would be more useful if it contained even a very short account of the theory of definite quadratic forms. It would place in the proper setting and correlate with the general theory such questions as the representation of a number as a sum of two or four squares. They give as many as four different methods, all interesting, for two squares and three for four squares, but these perhaps tend to shift the emphasis away from what is a really important section of number theory. It may be remarked that other proofs with more generality are available. There is the method of Thue applying to the representation of primes by forms  $x^2 + dy^2$  for several values of  $d$ , for example 1, 2, 3, 5, 7, etc. There is also a great deal to be said for giving the really elementary but general method of Uspenski (based upon Liouville's ideas) which easily proves formulae for the number of representations of an integer as the sum of  $2r$  squares for the smaller values of  $r$ .

It is hardly proper to speak of intentional omissions in this book, but the subject matter and the notes would be richer if, in dealing with congruences, there had been mention of some exceedingly interesting recent developments. In considering elementary congruences, or again in the representation of an integer as the sum of four squares, there occurs the congruence

$$x^2 + y^2 + 1 \equiv 0 \pmod{p}, \text{ or say, } f(x, y) \equiv 0 \pmod{p},$$

where  $p$  is a prime. It is easy to prove as they do that the congruence has at least one solution. Thus if  $p$  is odd, there are exactly  $\frac{1}{2}(p-1)$  quadratic non-residues of  $p$ , that is, integers  $n$  such that  $\zeta^2 \equiv n \pmod{p}$  is not solvable for  $\zeta$ . But as  $-y^2 - 1$  assumes  $\frac{1}{2}(p+1)$  different residues for the  $p$  values  $y = 0, 1, 2, \dots, p-1$ , at least one of them cannot be a quadratic non-residue and the result follows. It is not difficult to find the number  $N$  of solutions of the congruence, namely

$$N = p - (-1)^{\frac{1}{2}(p-1)},$$

which is not given in the book but might well have been. But when  $f(x, y)$  is a more general polynomial, the question becomes a very difficult one; for example, it may be difficult to establish the existence of even one solution. When  $p$  is a large prime, approximate formulae may be found in some cases. The order of magnitude of the error term is intimately associated with the zeros of polynomials having many of the characteristic properties of the Riemann zeta function, including an analogue of the famous Riemann hypothesis concerning its zeros. Thus it has been proved by Hasse by abstruse arithmetical methods that for the congruence

$$y^2 \equiv 4x^3 - ax - b \pmod{p},$$

where the discriminant  $a^3 - 27b^2 \not\equiv 0 \pmod{p}$ , the number  $N$  of solutions satisfies  $|N - p| \leq 2\sqrt{p}$ . This statement is equivalent to the assertion that

a certain analogue of the zeta function has its zeros on the critical line. But, as has been remarked, the authors do not mention the zeros of the Riemann zeta function and so perhaps one can hardly expect them to mention the problem above. Still, as they give a brief account of the error terms in the approximation to some arithmetical sums, and give further details about the divisor problem in their notes, it would have been interesting to mention a striking problem where the error order is known, a not too frequent occurrence in number theory.

L. J. MORDELL.

**Einführung in die Zahlentheorie.** By A. SCHOLZ. Pp. 136. R.M. 1.62. 1939. Sammlung Göschen, 1131. (Walter de Gruyter, Berlin)

This is an introduction in the more proper sense of the word. The author gives a strictly logical development of the elements of the classical theory, using, however, modern ideas and modes of expression when they are appropriate. He also, when it fits in with his scheme, gives some more advanced results not usually found in elementary books but none the less most welcome in enlarging the reader's horizon.

The book contains six chapters which may be briefly summarised as follows. The first deals in the rigorous manner now familiar in books on foundations of modern algebra with the arithmetic of the natural integers, starting from their definition as an ordered set with certain properties. This is expounded in a way rather unusual in introductory books, which often pass over the subject in silence. The second chapter is concerned with integers and their divisibility properties. It includes an account of moduli of integers, of the highest common factor and the least common multiple; of the Euclidean algorithm and the fundamental theorem of arithmetic on the unique factorisation of integers; the more elementary properties of prime numbers and some of the simpler arithmetical functions.

The third chapter gives a full account of the elementary properties of congruences presented from the modern viewpoint. Thus it is stated and proved that the ring formed by the residues with respect to a prime is a field. There are applications to decimals, primitive roots and power residues. There is included a simple and useful theorem due essentially to Thue that "If  $p$  is a prime number and  $e, f$  are integers such that  $1 < e, f \leq p, ef > p$ , then every residue  $r \pmod{p}$  can be expressed in the form  $r \equiv 0$  or  $r \equiv \pm x/y$ , where

$$0 < x < e, \quad 0 < y < f."$$

This is applied later in the chapter to show that certain primes  $p$  can be expressed in the form  $x^2 + dy^2$  for  $d = 1, 2, 3, 5, 7, 13, 37$  with integers  $x, y$ ; in particular when  $p \equiv 1 \pmod{4}$  and  $d = 1$ . It is shown that every positive integer can be expressed as the sum of four integral squares and also that  $x^4 + y^4 = z^4$  is impossible in integers unless  $xyz = 0$ .

Chapter IV deals with quadratic residues and the law of quadratic reciprocity. Eisenstein's simplification of Gauss' third proof is given. The final step, namely the proof of the result

$$\sum_{x=1}^{\frac{1}{2}(p-1)} \left[ \frac{qx}{p} \right] + \sum_{y=1}^{\frac{1}{2}(q-1)} \left[ \frac{py}{q} \right] = \frac{1}{2}(p-1) \cdot \frac{1}{2}(q-1),$$

where  $p, q$  are different odd primes and the square brackets denote the integral part, is made obvious by an enumeration of the values of  $x, y$  with

$$1 \leq x \leq \frac{1}{2}(p-1), \quad 1 \leq y \leq \frac{1}{2}(q-1),$$

according as  $qx - py > 0$  or  $< 0$ . The chapter ends with a useful sketch of the

ideas associated with the laws of cubic and biquadratic reciprocity and a proof of the biquadratic character of 2.

The fifth chapter is devoted to the theory of binary quadratic forms. It includes an account of the representation of numbers by them, of their reduction to a standard reduced form and of their automorphs. It also contains a sketch of the theory of composition and genera.

The final chapter is mainly a numerical one concerned with algorithms for finding or testing numerical results. Applications are given to factorisation, simultaneous linear congruences, tables and enumeration of prime numbers, and indices and solutions of congruences. The illustrations are often connected with suggestive results in number theory.

It is quite clear that the author has produced an excellent little book. It contains a great deal in a small compass but does not suffer in any way from condensation. It can be strongly recommended to all desirous of a cheap, convenient and rapid introduction into number theory which takes note of the influence of modern algebra upon the subject.

L. J. MORDELL.

**Elements of the topology of plane sets of points.** By M. H. A. NEWMAN. Pp. viii, 221. 12s. 6d. 1939. (Cambridge)

In recent years many books on topology have appeared, but none of them has been particularly suitable for those beginners who wish to master the fundamental definitions and techniques of the subject without losing themselves in a mist of extremely abstract and general ideas. Mr. Newman's book will meet their requirements. It is an excellent specimen of the English variety of textbook at its best. New notions are introduced step by step, and are well illustrated by examples at each stage, so that the student has time to familiarise himself with one idea before he encounters the next. There are a large number of problems for solution by the reader, and hints are given to make the hard ones easier.

The first part of the book contains an introduction to point-set topology in a general metric space, including the notions of homeomorphism and continuous mapping, of connection and local connection. It is shown how the simple arc and the simple closed curve can be characterised in a purely topological way. The second part is devoted to a discussion of combinatorial topology. The author restricts himself, in the main, to plane sets of points, but here and there he gives indications of how results can be extended to sets contained in Euclidean spaces of more than two dimensions or in even more general topological spaces. It will probably be useful to give here a more detailed summary of this part of the book.

Chapter V introduces the notion of complexes on a rectangular grating, and contains proofs of Alexander's lemma and its simpler consequences, such as Jordan's theorem that a simple closed curve determines two domains, one outside and one inside the curve. Here will also be found a proof of Brouwer's theorem on the invariance of open sets, viz., that if a set in  $p$ -dimensional Euclidean space is the topological image of an open set in the same space, it is itself an open set. Chapter VI is devoted to the subjects of simply-connected domains and deformability of paths. In particular it contains a proof of the fundamental theorem that every simply-connected domain in the plane is the topological image of the whole plane. The chapter concludes with an intellectually satisfying proof of Cauchy's theorem in the theory of analytic functions of a complex variable; here can be found a proper treatment of all the points that are slurred over in the ordinary analysis textbook. The next chapter is entitled "Accessibility and Jordan domains". The main questions here dealt with are the converse of Jordan's theorem and the

relation between local connectedness of a domain and local accessibility of the points of its frontier. Chapter VIII, which concludes the book, introduces the notion of connectivity numbers, and serves as a convenient jumping-off place for the many readers who will wish to pursue the subject further.

There are two misprints that it may be useful to mention; on page 92, 5 lines from the bottom, "even" should be "odd", and on page 189, line 5, "connectivity  $p$ " should be "connectivity  $p-1$ ".

There is a fairly good index, which, however, is not as complete as it might be; for instance, "enumerable" and "residual domain" are missing. One or two notions, including the important one of separation, do not seem to have received explicit definitions, and a few of the proofs might be made a little more formal with a slight gain in clarity. These, however, are minor blemishes, which subtract little from the value of the book as a whole. It is to be hoped that, as one result of its appearance, more attention will be paid to the subject of topology in the mathematics departments of British universities.

F. S.

**The Mathematical Theory of Non-uniform Gases.** By S. CHAPMAN and T. G. COWLING. Pp. xxiii, 404. 30s. 1939. (Cambridge)

This is a book on the kinetic theory of gases. Only close study would enable one to pass a good judgment on its details, so we shall (to use a euphemism of Macaulay's) be inclined to think we shall best meet the wishes of readers by a more general survey. Reviewers, by immemorial custom, perform their task by copying out the preface. Here we look less exhaustively at the fly-leaf, where the first pleasant feature awaits us, for the book is dedicated to David Enskog.

To appreciate the advantages of this book we should think of what there was to read before it was published. We can leave on one side the student who, after a struggle, is content to divide the molecules into six streams moving parallel to the edges of a cube. Those who penetrate deeper will know that a more elaborate, but still elementary theory presents difficulties, like the persistence of velocities, which lead them on till they wonder if it is ever going to end. How it does end is shown in this book, but before it came out the student might have read Boltzmann's *Gastheorie* or Jeans' *Dynamical Theory of Gases*. Boltzmann appeared over forty years ago. Unrivalled in its day, its main interest lies now in its accessible account of Boltzmann's famous equation. When it was published the theories of viscosity and thermal conduction seemed hopelessly difficult. But a book by a discoverer is always interesting, and this is no exception. Jeans' book falls short of what one would expect from the times at which it was written. The first edition does not contain Boltzmann's equation, doubtless because it had no solved problem to its credit. Too much of the book consists of a discussion of the assumption of molecular chaos. The second edition, published after Chapman's first paper, was still largely an account of the elementary theory perfected as much as possible. Chapman's papers are not simple. Without going so far as to say (to modify another of Macaulay's phrases) that no one whose heart is less stout than that of a compositor has read them, one may still think them harder than Enskog's, and appreciably less attractive: Enskog's dissertation of 1917 contained most of Chapman's results obtained afterwards in ignorance of Chapman's work. With Enskog at last we have a theory which is reasonably simple and easy to grasp, the long-sought solution of Boltzmann's equation, rigorous enough though not proved to converge. It might have had full treatment in the third edition of Jeans. We find, not an exposition of the extent and degree of digestion appropriate to a first-class textbook, but

the kind of reference which one may see in those books on the quantum theory in which the reader is just beginning to be interested in some striking theoretical achievement when the harpies take away his prospective meal and give him the result and the statement that Sommerfeld proved it.

Thus until recently a really serious student of the kinetic theory (by whom we mean one who approaches it in the hope of research though he may afterwards find it has all been done) would have to get hold of Enskog's dissertation. This can be done by writing to Upsala; but who knows it? At least one researcher has said that his knowledge of Enskog is not first hand.

These difficulties are removed by the welcome appearance of Chapman and Cowling, who have come down from the stars for the purpose. If we were to prescribe a course of reading for our serious student, it would be to begin with chapters 7 to 10 and look back for what he needs. In about a hundred pages he will find the modern theory expounded fully and accurately, and as simply as the subject allows. There are no free paths. Everything depends on a single statistical assumption, that collisions at all points of the cross-section of the cylinder of molecular action are equally probable. If this, the assumption of molecular chaos, is allowed everything follows, in theory at least. This is the idea of Maxwell's great memoir of 1866, which had only to be supplemented by Boltzmann's equation (in the same field of ideas) to reduce the whole problem to one of mathematics. The mathematics is not easy, and has taken seventy years to perfect. Some jib at it and hanker after their free paths; but the history of the science proves conclusively that they fall into error or uncertainty great or small.

The questions one would naturally ask about the modern theory are answered incidentally by Chapman and Cowling: how the elementary theory is defective, why there is sometimes no great harm in a restriction of the molecule to certain models and to spherical symmetry, when quantum mechanics will appreciably affect the results of Newtonian mechanical analysis, and so on. Physicists will welcome chapter 18, which puts electromagnetic phenomena in ionised gases in the general theory, and the extensive comparisons between theory and experiment in chapters 12 to 14. The book will be for a long time the last word on a theory that has exercised so many. The authors describe Burnett's work as probably the final improvement to be made in the general theory of a non-uniform gas at ordinary temperatures. The polynomials of Sonine used by him are a generalisation of those introduced by Laguerre a year before, and have since become of great importance in the theory of the hydrogen atom in quantum mechanics. The confluent hypergeometric function and the corresponding polynomials have become better known in recent years, and are now an essential part of the equipment of a mathematical physicist.

F. B. P.

**Vector Methods.** By D. E. RUTHERFORD. Pp. viii, 127. 4s. 6d. 1939. University mathematical texts, 5 (Oliver and Boyd)

The theory of vectors occupies two chapters, "Vector algebra" and "The vector operator nabla". There are also chapters showing applications to differential geometry, mechanics, potential theory, and hydrodynamics, together with a short but interesting account of four-dimensional vectors. The chapter on Laplace's equation is rather out of place as it contributes nothing to vector theory, nor do several of the books mentioned in the bibliography at the end.

With regard to the treatment Dr. Rutherford uses the dot and cross notation for products, and founds the work mainly on cartesian coordinates. The magnitude of a vector is called the "length", which leads to the rather odd

statement at the top of p. 19 that the length of a certain vector is  $1/\rho$ , where  $\rho$  is the radius of curvature.

One cannot feel entirely pleased with the device of giving unrelated definitions of *div*, *curl*, and *grad*, nor with the absence of mention of the various aspects of the theorems of Stokes and Gauss according to the form of the integrand. The statement on p. 68 that the vanishing of the curl implies the vanishing of the circulation may be false in the case of an irreducible circuit.

The chapter on differential geometry is very pleasing for the briskness of the style. Incidentally on p. 17 a second order term is omitted from the vector  $OP_1$  and consequently from the vector  $OP_2$ .

The book certainly gives a lot of useful matter in a small compass and may be held to fulfil the object stated in the Preface "to provide a clear account of the abstract theory and a brief but broad survey of the application of the theory to various branches of pure and applied Mathematics". L. M. M.-T.

**Mathematical Tables. VII. The Probability Integral.** By W. F. SHEPPARD. Completed and edited by the British Association committee for the calculation of mathematical tables. Pp. xi, 34. 8s. 6d. 1939. (Cambridge University Press, for the British Association)

For many years Dr. Sheppard had in hand the preparation of tables of the probability integral to give that function to as many decimal places as were ever likely to be required. Shortly before his death in 1936 his manuscripts were placed at the disposal of the British Association committee for the calculation of mathematical tables, who have completed the work.

The probability integral was first tabulated by Kramp, in 1789, and afterwards by Bessel, Legendre, Encke, De Morgan, and Glaisher. The best of the older tables was due to Burgess (1898). All these used multiples of the modulus as the tabular argument. Sheppard was the first to use multiples of the standard deviation. His first table appeared in 1899, and was followed by a much better one in 1902-3. Another, due to him and Galton, was published in 1906-7. These three tables were included in *Tables for Statisticians and Biometricians*, Part I (1st ed. 1914, 2nd ed. 1924). A number of other tables, mostly less extensive, have been published as appendices to textbooks on statistics; some of these appeared in an improved and extended form as the *Kelley Statistical Tables* (1938). Part II of *Tables for Statisticians and Biometricians*, which appeared in 1931, contained three tables; Sheppard's unpublished work was used to assist in their computation.

The present volume of tables contains six tables, all up to ten times the standard deviation. Some idea of the improvement on existing work may be given by pointing out that Table VI gives seven significant figures correctly over the whole range, whereas the best of the previously published tables, although very accurate in some places, in others could not be depended upon for more than two significant figures.

H. T. H. P.

**Étude critique de la Notion de Collectif.** By J. VILLE. Pp. 144. 75 francs. 1939. Monographies des Probabilités, Calcul des Probabilités et ses applications, publiées sous la direction de M. Émile Borel, Fascicule III. (Gauthier-Villars, Paris)

The term *collective* is due to von Mises, who defined it as a mass phenomenon or an unlimited sequence of observations fulfilling two conditions: (i) the relative frequencies of particular attributes of single elements of the collective tend to fixed limits; (ii) these fixed limits are not affected by any place selection. M. Ville has made a critical study of the ideas of von Mises himself, and also of the variations of them proposed by Reichenbach, Popper, Cope-

land, and Wald. After a preliminary chapter on general principles, the author devotes five chapters to a detailed study of the subtle concepts involved in the idea of a collective. These chapters are not particularly easy reading, but perhaps this is because of the inherent difficulties of the subject. M. Ville is constructive as well as critical. His contributions have been commended by Professor M. Fréchet, who, after mentioning the propositions established by M. Ville, says that he seems to have approached more nearly to the object of von Mises' aims than any one else so far. But M. Ville is not satisfied with von Mises' definition, either in its original form or in any of its modifications. In a final chapter, which is much easier reading than the preceding five, he raises four objections. The first is that the new theory excludes certain aggregates admissible in the classical theory. However, as it is very doubtful whether these aggregates have any physical significance, we need not worry over their loss. The second difficulty is more serious. How can we tell if a long sequence forms part of a collective? Apparently one must wait for all eternity before giving an answer. In practice, no doubt, mortal men would jump to a conclusion after a much shorter period, but if so they would run the risk of being mistaken, especially if a long run of unusual results came early in the trials. The third objection is to the second part of the definition, concerning place selection. If one does not think about it too much, the meaning of this phrase appears obvious from von Mises' explanation, but many authors consider it a subjective notion which is impossible to define in mathematical terms. The fourth objection is that the new theory fails to attribute any measure of probability to an isolated event. However, an ingenious method of avoiding this difficulty has been pointed out by Reichenbach.

M. Ville's final conclusions are that he agrees with von Mises in the opinion that the theory of probability has no objective value except when applied to an extensive group of events. They part company when it comes to defining what groups are admissible. M. Ville considers that any attempt at an exact definition, such as that of a collective, leads to the neglect of other groups which should not be excluded. It seems to him that we are forced to use phrases which are incapable of exact mathematical definition, such as "the frequencies have a tendency to be grouped round a certain value". Any attempt to define this tendency as the approach to a limit, in the strict mathematical sense, either leads to results which are inconsistent with themselves, or, at best, needs an elaborate array of artificial restrictions which seem in no way imposed by the real nature of the problem. H. T. H. PIAGGIO.

**Differentialgeometrie der Kurven und Flächen und Tensorrechnung.** By V. HLAVATÝ, translated by M. PINL. Pp. xi, 569. Fl. 14; geb. fl. 15.50. 1939. (Noordhoff, Groningen)

This work is a systematic treatise on differential geometry in ordinary Euclidean space of three dimensions. Its scope is roughly that of the existing standard treatises on the subject, but it makes extensive use of modern methods and uses throughout vectors and tensors, the necessary properties of these being developed in the text. The treatment of curves follows the conventional lines, though with more precision than in many works on the subject. When we come to the theory of surfaces the conventional order of treatment is somewhat modified. A very extensive chapter, amounting to some two hundred pages, is devoted to those properties of surfaces which depend only on the first fundamental form: conformal representation, geodesics and the theorems of Gauss and Bonnet. Only then is the second fundamental form introduced, leading to a study of lines of curvature, asymptotic lines, and so on. A final chapter discusses particular types of surfaces (ruled, minimal, etc.).

This book may be warmly recommended; the text is precise and clear, and the order of development of the subject is logical. The printing is beautifully done, and there seem to be comparatively few misprints even in the most complicated formulae. J. A. TODD.

**Advanced Analytic Geometry.** By ALAN D. CAMPBELL. Pp. x, 310. 20s. 1938. (John Wiley and Sons; Chapman and Hall)

It is difficult to place this book within the scheme of the teaching of geometry in England. In some ways the outlook is advanced and appropriate for a university student; yet in other respects ignorance is assumed, and not always remedied, about matters commonly dealt with in the sixth forms of our schools.

Part I (Chapters 1 to 8) is an introduction to affine plane analytic projective geometry. After some preliminary work about rectangular and oblique axes and transformations, the transformation  $x = a_1x' + a_2y' + a_3$ ,  $y = b_1x' + b_2y' + b_3$  is studied in Chapter 3, which contains some rather sketchy work on matrices, products of transformations, and invariants. Then cross-ratio is introduced. Incidentally it is stated that the Greeks called a harmonic set of four collinear points the golden section. A "first hint" of infinite points is not satisfactory since the "peculiar situation" is interpreted by saying that a certain line meets another in a so-called infinite point. The discussion of imaginary elements in Chapter 6 is also unsatisfactory. Chapter 5 introduces groups of linear transformations, their invariant points and lines, and the associated geometries. Chapter 7 gives some properties of curves of order  $n$ ; it is curious that, of the ten sections of this chapter, asymptotes, tangents, and the line at infinity should occupy the fifth, seventh, and tenth respectively. Chapter 8 deals with conics and linear systems of conics.

There are some interesting numerical examples, and the exercises usually contain questions of the form "check the algebraic details and answer all the queries (Why?) in the text", as well as more ordinary exercises. The technique is not good; as a trivial example of this the author prefers to represent a net of conics by  $\lambda C_1 + \mu C_2 + C_3$  and to say that to count  $C_1 = 0$  as a member of the net we must allow  $\lambda$  to take the value  $\infty$ , instead of representing the net by  $\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3$ .

Part II (Chapters 9 to 18) is an introduction to general plane analytic plane geometry. A cartesian framework is first set up from arbitrary points (1, 0), (0, 1) by using parallels. Then, with the help of Desargues' theorem and some properties of the quadrangle, a triangle of reference is defined by a generalisation of the above process. So far, a point has had only two coordinates, but now homogeneous coordinates are introduced, and properties of the circular points at infinity, homogeneous line-coordinates, and plane duality are given. It is explained, but not proved, that the geometry obtained could have been built up from certain assumptions in Veblen and Young's *Projective Geometry*. At the end of Chapter 12, it is explained briefly how the subject could have been developed by starting with points  $(x, y, z)$  and lines  $ux + vy + wz = 0$ .

Chapter 13 deals with the general linear transformations of point and line coordinates, and their invariant points and lines. The next chapters are concerned with correlations and polarities, conics, and quadrangles; the last two chapters contain further properties of curves of order  $n$  and linear systems of conics.

From the English point of view, Part II seems more satisfactory than Part I, because it deals with topics less likely to be covered in the elementary course. The emphasis on the ideas connected with transformations is a noticeable feature. It is unlikely, however, that the English student will find the book well adapted to meet his needs at any particular stage of his career. A. R.

**An Introduction to Modern Geometry.** By L. S. SHIVELY. Pp. xi, 167. 10s. 1939. (John Wiley, New York; Chapman and Hall)

This is an attractive book on the limited range of geometry which it covers. It may be well to mention these limitations first. There is no projection, either orthogonal or general, though the distinction between metrical and projective geometry is explained, after the proof of Desargues' theorem from Menelaus' theorem. There is nothing about conics. Reciprocation is not dealt with, though the principle of duality is explained. Inversion is discussed and the bookwork includes Peaucellier's cell and a proof of Feuerbach's theorem; there are, however, only a few examples of theorems to be proved by inversion.

There are chapters on similar figures, the theorems of Ceva and Menelaus, harmonic points and lines, the triangle, coaxial circles, inversion, poles and polars, cross-ratio, involution. In all these the bookwork is set out in a clear and readable manner and there are adequate sets of examples mainly of a straightforward type, well adapted to bring home the consequences of the theorems in the text.

There is then an interesting chapter on "constructions with ruler and compass"—the instruments not only being taken together but also separately and constructions with a ruler and fixed circle being included. The simplicity and exactness of constructions are also discussed and Lemoine's criteria stated. Then there is a final chapter on "selected theorems and problems"; an interesting selection is made, and a set of 100 miscellaneous supplementary exercises is given.

Teachers should find this book a useful one and it is thoroughly suitable for the pupil's use, provided the limitation of the range covered is not regarded as unsatisfactory.

C. O. T.

**The Collected Works of George Abram Miller. II.** Pp. xi, 537. \$7.50. 1938. (University of Illinois Press, Urbana, Ill.)

The present volume consists in the main of Professor Miller's contributions to the theory of groups of finite order during the years 1900-1907. Included also are two recent papers, a "Note on the history of group theory during the period covered by this volume" and an article entitled "Primary facts in the history of mathematics". Not all of Miller's papers published in the years mentioned above have been included. From those whose interest is primarily didactic a selection has been made by the Editorial Committee.

The years in question saw the interest of European group-theorists largely concentrated on the subject of linear groups, and more especially on the theory of the representation of abstract groups by linear groups, associated chiefly with the names of Frobenius, Burnside and Schur. In these developments, Miller took little part. And the papers in the present volume, over a hundred in number, for the most part concern such topics as groups of automorphisms, groups of prime-power order and groups defined by a set of generating relations, in all of which domains Miller was a pioneer.

The printing and binding leave little to be desired, and the editing and indexing seems to have been carefully done, and fully maintain the high standard already set in the first volume.

P. HALL.

**A Preface to Mathematics.** By C. E. VAN HORN. Pp. xii, 124. \$2.50. 1938. (Chapman and Grimes, Boston)

In a very readable little book Dr. Van Horn seeks to provide a helpful background for the teacher of Mathematics. With this end in view he takes a rapid survey of the field of operations, dwelling for a space on such topics as

present special difficulties to the pupil or serve to illustrate some general principle. For an outline sketch it is surprising how much has been included in the way of historical detail, *obiter dicta* and other interesting matter.

"Why study Mathematics?" asks the author. To answer the question he takes each branch of the subject in turn, Arithmetic, Algebra, Geometry, and so on, describing how it has developed in response to practical needs and giving specific instances of striking contacts. Some of these are distinctly novel; who would suppose that Euclid's Golden Section is used in shaping the printed page, or that the logarithmic spiral is to be found, not only in machine cams, but also in the elephant's tusks?

A more detailed examination follows. Most space is devoted to the Calculus, but other sub-divisions receive their due need. Functionality, partial differentiation, Taylor's theorem are dealt with and Riemann's definite integral discussed with some thoroughness.

Of misprints we have noticed only two, both occurring on p. 83, where dashes have been omitted.

Taken altogether the work is suggestive and stimulating, giving a perspective view which will certainly be of service to teacher and pupil alike.

H. L.

**Teaching the New Arithmetic. What to Teach. How to Teach It. Provision for Professional Growth.** By G. M. WILSON, M. B. STONE and C. O. DALRYMPLE. Pp. xi, 458. 18s. 1939. (McGraw-Hill)

This book fulfils very successfully the purposes indicated in its sub-titles. It is a very complete analysis and summary of the results of twentieth-century investigations into the methods and content of the arithmetic programme; and it may claim to encourage professional growth through its enthusiastic belief in the importance of the contribution which teachers can make to the study of common adult usage and the adaptation of teaching to the needs of individual pupils. The authors accept the necessity for educational diagnosis and remedial tuition, and they go further than most present-day practice in their limitation of the minimum essentials to the needs of the ordinary non-vocational public. To this they add suggestions for activities, for more advanced special varieties of computation, and for appreciation of the number system and its history.

The chapters on common fractions, decimal fractions, percentage, interest and measures bring together material not previously presented in book form; and in chapters XXII to XXVII a very good summary is given of the present position of research into written problems and into the frequency of arithmetical problem-situations in adult life.

Besides competent sectional references a good general bibliography is given under the headings: Curriculum, Methods of Teaching, Drill, Written Problems, Corrective Arithmetic, Games and Activities, Research in Arithmetic. For these bibliographies alone the volume is worth buying.

As a whole the book is somewhat lacking in emphasis on the importance of training in what has been called functional thinking in mathematics—the recognition, interpretation and utilisation of relationships—but it provides a most valuable summary of recent research, and it brings together evidence which should prove useful to those who are seeking to reduce the amount of unnecessary computation which is asked from pupils in the primary school.

C. M. F.

**Commercial Arithmetic.** By H. V. ALLEN. Pp. ix, 432. 6s. 1939. (Longmans, Green)

This *Commercial Arithmetic* is written for those who are studying for the

Intermediate and Advanced Examinations of the Royal Society of Arts and the London Chamber of Commerce, and for the Intermediate Examination of the Institute of Secretaries. Very elementary topics are omitted and there is an Appendix of 25 pages on Mensuration, which contains a collection of formulae, 8 worked examples, and 87 examples. It is preceded by a note on units which contains such incorrect statements as "A metre is  $10^{-7}$  of the distance between the pole and the equator", "The gramme is the weight of 1 c.c. of water at  $4^{\circ}\text{C.}$ ", "A litre is 1000 c.c.". Briefly, a litre is the volume of 1 kg. of distilled water at maximum density and normal pressure and is approximately 1000.03 c.c.

The various topics are carefully arranged, the explanations are clear and are illustrated by numerous worked examples. Each section closes with a full set of examples mainly selected from questions set at examinations. The book is not merely a collection of topics and examples. A distinguishing feature is the introductory note which precedes each topic. This explains Commercial Practice, is interestingly composed, and is not merely a few lines of scamped explanation. These explanatory notes occupy many pages of the text, and students are bound to derive much benefit from the interest and life they impart to the book.

They are unfortunately not free from errors. In the table on stamp duties for Bills of Exchange, it is stated that "for £25 and under £50, duty = 6d.". It should read "Over £25 and not exceeding £50, duty = 6d.". Thus for £50 the duty is 6d., not 9d. as stated by the author. The limits in the remaining statements in the table all have the same mistake. There are some errors in the chapter on Stocks and Shares. Commission for buying shares is not from  $1\frac{1}{2}\text{d.}$  per share. If the share is from 1s. to 2s. it is  $\frac{1}{2}\text{d.}$  per share. Again, the author would not get away with a commission of  $\frac{1}{8}$  per cent. if he bought Government Securities unless he bought £20,000 stock and even then only if they were redeemable within 12 years. In the worked example, p. 299, the Contract Stamp on £974 should be 2s., not 17s. 6d.

In the short section on contracted work there is an absence of any clearly stated principle. On p. 179 the author implies the following method for evaluating an example of the type  $\sqrt[n]{a/b}$  to  $n$  significant figures: "Work  $a/b$  to  $2n$  figures, etc." Actually it is quite safe to work  $a/b$  to only  $(n+1)$  figures. This follows from the fact that any relative error due to approximating  $a/b$  is halved in the subsequent square root. The Troy Weight table on p. 17 is completely wrong.

There are several useful interest tables and a set of 5-figure logarithms.

S. INMAN.

**A Scottish School Geometry, Parts II and III.** By W. G. THOMSON. Pp. 247-324, 325-428. 1s. 6d.; 1s. 9d. 1937. (Oliver and Boyd)

These two volumes are the second and third parts of a four-part work written especially for Scottish schools, and as they are concerned with Ratio and Proportion and Rectangles and Squares, they apply to those pupils who are candidates for the Leaving Certificate Examinations. There are 58 theorems and 582 exercises, so that—quantitatively—there is value for the money. But there is one aspect at least in which the quantity given rather defeats the aim of the author, and that is in what may be called the "continuous narrative" style that he adopts. In order to make clear what is meant by this, it is necessary to quote verbatim from the book. For example on page 281 is found: "Since  $CF$  and  $AD$  are parallel, the angle  $CGB$  is equal to the corresponding angle  $ADB$ , which is equal to the angle  $ABD$ , for  $AB$  is equal to  $AD$ . Therefore the angle  $CGB$  is equal to the angle  $CBG$ , and there-

fore, in the triangle  $CGB$ , the side  $CB$  is equal to the side  $CG$ ; that is in the rectangle  $CK$ , a pair of adjacent sides are equal. Therefore the rectangle  $CK$  is a square, namely, the square described on  $CB$ ." Surely this is considerably more effective when written :

$$\begin{aligned}\angle CGB &= \angle ADB \text{ (corr.)} \\ &= \angle ABD \text{ (} AB=AD \text{)} \\ &= \angle CBG ;\end{aligned}$$

$$\therefore CB=CG \text{ (isos. } \triangle \text{)} ;$$

$\therefore$  the rect.  $CK$  is a square.

The constructions that are given in the geometrical proofs of the identities

$$(a+b)^2 = a^2 + 2ab + b^2, \quad (a-b)^2 = a^2 - 2ab + b^2, \quad (a+b)(a-b) = a^2 - b^2$$

give me the impression that they are especially designed to trap the unwary, which presumably is *not* what the author intended.

In reviewing a Geometry book intended for candidates in the Leaving Certificate Examinations the attitude of the Scottish Education Department towards the subject must be remembered. During the last decade the Department has been gradually descending from its strictly purely-Euclidean pedestal, until now there are only two or three steps down which it may go until it reaches the plane of ordinary man. For many years controversy raged over "Triangles (and parallelograms) of the same altitude are proportional to their bases", and finally a compromise was effected. The Department adhered to its principle that it would not recognise any proof of a geometrical theorem that depended on a mensuration formula, but that it would not ask for a formal proof of this particular theorem. The reason for this condescension is possibly that even the Department agrees—unofficially of course!—that the "natural" way to prove this theorem is to quote the formula for the area of a triangle (or parallelogram). Consequently it is with surprise that one finds the old-time proof given on page 338, although it is not valid for incommensurables, and so there seems no reason, either mathematical or pedagogical, for its inclusion.

In view of these two major criticisms I feel that I cannot recommend Mr. Thomson's book except to those who like formalism: in fact the general treatment gives the impression that his motto has been "Back to Euclid at any price". Personally for school use I prefer less formalism and more breeziness, which may be rank heresy but is certainly more effective. Also Euclid may be an unconscionable time o' dying but there are signs that a demise is not very far away.

A. I.

#### CORRECTION.

**Mathematics for Actuarial Students.** By H. FREEMAN. (Cambridge University Press)

We regret that by an oversight the name of the author was omitted from the heading of the review of this book (*Gazette*, XXIII, October 1939, p. 423) and offer our apologies to the author and publishers for the omission.

# CARDIFF BRANCH REPORT FOR 1937-1938.

THE Branch has had a very successful year under the Presidency of Mr. D. Vaughan Johnson, Chief Inspector of the Central Welsh Board.

Four meetings were held during the session. The retiring President (Mr. A. Buxton) delivered his address on "Heights and distances", dealing with the practical problem of ranging on aircraft. Mr. E. J. Atkinson demonstrated some of his ingenious apparatus for practical courses in applied mathematics. Dr. Neumann read a paper on Topology; and the concluding meeting was addressed by Mr. C. O. Tuckey, whose subject was the new Geometry Report of the Association. This meeting was advertised throughout the schools in South Wales and the room was packed out by a very appreciative audience.

A. HEDLEY POPE.

## LONDON BRANCH.

At a meeting on December 10th, 1938, Mr. C. W. Stokes gave an address on "The teaching of trigonometry in the main school". The paper emphasised throughout the importance of introducing trigonometry as early as possible into the school course and of applying it whenever opportunity arose in later work. As soon as scale drawing was mastered, the idea of the tangent of an angle could be given, and a little later those of the sine and cosine. At this stage field-work with simple instruments should be done. Numerous instances were given of theorems in geometry which could be interpreted by trigonometry, thus showing that a result which could be obtained by construction could also be reached by calculation. The application of trigonometry to three-dimensional problems was shown and the suggestion made that some spherical trigonometry might easily be introduced by a discussion of great circle sailing.

A. J. TAYLOR.

## THE INTERNATIONAL CONGRESS OF MATHEMATICIANS.

CAMBRIDGE, MASSACHUSETTS, U.S.A., SEPTEMBER 4-12, 1940.

ON invitation by the American Mathematical Society, an International Congress of Mathematicians will be held in Cambridge, Massachusetts, in 1940.

Detailed information will be sent in due course to all members of the American Mathematical Society. Others interested in receiving information may file their names in the Office of the Society, and such persons will receive from time to time information regarding the programme and arrangements.

Communications should be addressed to the American Mathematical Society, 531 West 116th Street, New York City, U.S.A.

## BOOKS RECEIVED FOR REVIEW.

**P. B. Ballard.** *The ABC of algebra.* Pp. 128. Limp cloth, 1s. 7d.; paper, 1s. 5d. 1938. (University of London Press)

**W. D. Baten.** *Elementary mathematical statistics.* Pp. x, 338. 15s. 1938. (John Wiley, New York; Chapman and Hall)

**R. J. T. Bell.** *Coordinate solid geometry.* (Chapters I-IX of *An elementary treatise on coordinate geometry of three dimensions.*) Pp. xiii, 175, xliii. 7s. 6d. 1938. (Macmillan)

**L. M. Blumenthal.** *Distance geometries.* Pp. 145. \$1.25. 1938. University of Missouri Studies, XIII, 2. (University of Missouri)

**F. Bowman.** *Introduction to Bessel functions.* Pp. x, 135. 10s. 6d. 1938. (Longmans, Green)

**E. Cartan.** *Leçons sur la théorie des Spineurs. I. Les Spineurs de l'espace à trois dimensions.* Pp. 98. 25 fr. 1938. Actualités scientifiques et industrielles, 643; exposés de géométrie, IX. (Hermann, Paris)

**W. S. Catto and F. J. H. Williams.** *An introduction to co-ordinate geometry. The straight line and circle.* Pp. 212. With answers, 3s. 6d.; without answers, 3s. 1939. (Harrap)

**J. Cavailles.** *Remarques sur la formation de la théorie abstraite des ensembles. I. Préhistoire. La création de Cantor.* Pp. 107. 25 fr. *II. Dedekind. Les axiomatisations (Zermelo, Fraenkel, Von Neumann).* Pp. 97-145. 12 fr. 1938. Actualités scientifiques et industrielles, 606, 607; le progrès de l'esprit, VII, VIII. (Hermann, Paris)

**J. Cavailles.** *Méthode axiomatique et Formalisme. I. Le problème du fondement des mathématiques.* Pp. 75. 20 fr. *II. Axiomatique et système formel.* Pp. 76-124. 18 fr. *III. La non-contradiction de l'arithmétique.* Pp. 125-194. 18 fr. 1938. Actualités scientifiques et industrielles, 608, 609, 610; le progrès de l'esprit, IX, X, XI. (Hermann, Paris)

**C. J. Coe.** *Theoretical mechanics. A vectorial treatment.* Pp. xiii, 555. 21s. 1938. (Macmillan)

**L. J. Comrie.** *Tables of  $\tan^{-1}x$  and  $\log(1+x^2)$ .* Pp. 18. 3s. 9d. 1938. Department of Statistics, University College, London; Tracts for Computers, XXIII. (Cambridge)

**H. S. M. Coxeter, P. Du Val, H. Flather, and J. F. Petrie.** *The fifty-nine icosahedra.* Pp. 26; XX plates. 4s. 6d. 1938. University of Toronto Studies, mathematical series, 6. (University of Toronto Press; Humphrey Milford)

**A. Dakin and R. I. Porter.** *Elementary analysis.* Pp. x, 315, xl. 6s. 1938. (Bell)

**P. Dienes.** *Logic of algebra.* Pp. 76. 18 fr. 1938. Actualités scientifiques et industrielles, 614; logique et méthodologie, III. (Hermann, Paris)

**F. Enriques.** *Le matematiche nella storia e nella cultura.* Pp. 339. L. 20. 1938. (Zanichelli, Bologna)

**P. B. Fischer.** *Arithmetik.* Pp. 152. RM. 1.62. 1938. Sammlung Götschen, 47. (Walter de Gruyter, Berlin)

**R. A. Frazer, W. J. Duncan and A. R. Collar.** *Elementary matrices and some applications to dynamics and differential equations.* Pp. xvi, 416. 30s. 1938. (Cambridge)

**A. Geary, H. V. Lowry and H. A. Hayden.** *Mathematics for technical students. I.* Pp. viii, 314. 4s. 1938. (Longmans, Green)

**S. H. Glenister.** *Rational geometry for schools. I. II. III.* Pp. 62; 64; 64. 1s. 6d. 1938. (Harrap)

**H. G. Green.** *School Certificate geometry.* Pp. xiv, 239. 4s. 6d. 1938. (English Universities Press)

**M. Hachtroudi.** *Les Espaces d'éléments à connexion projective normale.* Pp. 85. 20 fr. 1937. Actualités scientifiques et industrielles, 565. (Hermann, Paris)

**G. H. Hardy and E. M. Wright.** *An introduction to the theory of numbers.* Pp. xvi, 403. 25s. 1938. (Oxford)

**O. Haupt und G. Aumann.** *Differential- und Integralrechnung. I. Einführung in die reelle Analysis.* Pp. 196. RM. 11.20. *II. Differentialrechnung.* Pp. 168. RM. 9.80. *III. Integralrechnung.* Pp. 183. RM. 10.60. 1938. Götschens Lehrbücherei, Reihe I, 24, 25, 26. (Walter de Gruyter, Berlin)

**T. Herberg.** *Answers to "Elementary mathematical analysis".* Pp. 10. 6d. 1938. (D. C. Heath, Boston; Harrap)

- G. Hoheisel.** *Gewöhnliche Differentialgleichungen.* 3rd edition. Pp. 126. RM. 1.62. 1938. Sammlung Göschel, 920. (Walter de Gruyter, Berlin)
- H. W. Holtappel.** *Tafels van  $e^x$ .* Pp. xxxi, 132. Fl. 6. 1938. (Noordhoff, Groningen)
- V. F. Hopper.** *Medieval number symbolism.* Pp. xii, 241. 15s. 1938. (Columbia University Press; Oxford University Press)
- G. Hostelet.** *Les Fondements expérimentaux de l'Analyse mathématique des faits statistiques.* Pp. 70. 15 fr. 1937. Actualités scientifiques et industrielles, 552; le progrès de l'Esprit, II. (Hermann, Paris)
- G. Hostelet.** *Le Concours de l'Analyse mathématique à l'Analyse expérimentale des faits statistiques.* Pp. 70. 15 fr. 1938. Actualités scientifiques et industrielles, 585; le progrès de l'Esprit, III. (Hermann, Paris)
- M. Huber.** *Cours de Démographie et de Statistique sanitaire. I. Introduction à l'étude des statistiques démographiques et sanitaires.* Pp. 67. 15 fr. *II. Méthodes d'élaboration des statistiques démographiques (Recensements, État civil, migrations).* Pp. 110. 20 fr. 1938. Actualités scientifiques et industrielles, 598, 599. (Hermann, Paris)
- G. Kowalewski.** *Magische Quadrate und magische Parkette.* Pp. 78. RM. 2. 1937. Scientia delectans, 2. (Kochler, Leipzig)
- G. Kowalewski.** *Der Keplersche Körper und andere Bauspiele.* Pp. 65. RM. 2. 1938. Scientia delectans, 3. (Kochler, Leipzig)
- G. Kowalewski.** *Grundbegriffe und Hauptsätze der höheren Mathematik insbesondere für Ingenieure und Naturforscher.* Pp. 156. RM. 5. 1938. (Walter de Gruyter, Berlin)
- A. Lautman.** *Essai sur l'unité des Sciences mathématiques dans leur développement actuel.* Pp. 60. 15 fr. 1938. Actualités scientifiques et industrielles, 589; le progrès de l'Esprit, IV. (Hermann, Paris)
- A. Lautman.** *Essai sur les notions de structure et d'existence en mathématiques. I. Les schémas de structure.* Pp. 82. 20 fr. *II. Les schémas de genèse.* Pp. 83-162. 20 fr. 1938. Actualités scientifiques et industrielles, 590, 591; le progrès de l'Esprit, V, VI. (Hermann, Paris)
- N. Lazar.** *The importance of certain concepts and laws of logic for the study and teaching of geometry.* Pp. 65. \$1. 1938. (The Mathematics Teacher, New York)
- LL. S. Lloyd.** *A musical slide-rule.* Pp. 25. 2s. 1938. Slide-rule separately, 1s. (Oxford University Press.)
- G. A. Miller.** *The collected works of George Abram Miller. II.* Pp. xi, 537. \$7.50. 1938. (University of Illinois Press)
- L. M. Milne-Thomson.** *Theoretical hydrodynamics.* Pp. xxviii, 552. 31s. 6d. 1938. (Macmillan)
- F. D. Murnaghan.** *The theory of group representations.* Pp. xi, 369. 22s. 6d. 1938. (The Johns Hopkins Press, Baltimore; Humphrey Milford)
- W. G. Points.** *The young citizen's arithmetic, I-III.* Pp. 80, xvi; 80, xii; 88, xii. 1s. 6d. each; without answers, 1s. 3d. each. 1938. (Oxford University Press)
- H. W. Reddick and F. H. Miller.** *Advanced mathematics for engineers.* Pp. x, 473. 20s. 1938. (John Wiley, New York; Chapman and Hall)
- A. F. Robinson.** *Constructive arithmetic. I-IV.* Pp. 72, 8; 72, 9-20; 80, 21-31; 96, 32-47. I, II: 1s. 4d. each; without answers, 1s. 2d. each. III: 1s. 6d.; without answers, 1s. 3d. IV: 1s. 6d.; without answers, 1s. 4d. 1938. (Oxford)
- T. G. Room.** *The geometry of determinantal loci.* Pp. xxviii, 483. 42s. 1938. (Cambridge)

G. Scheffers. *Lehrbuch der Mathematik zum Selbstunterricht und für Studierende der Naturwissenschaften und der Technik*. 7th edition. Pp. viii, 743. RM. 15. 1938. (Walter de Gruyter, Berlin)

J. A. Schouten and D. J. Struik. *Einführung in die neueren Methoden der Differentialgeometrie. II. Geometrie*. 2nd edition. Pp. xii, 338. Geh. fl. 11.50; geb. fl. 12.50. 1938. (Noordhoff, Groningen)

J. F. Scott. *The mathematical work of John Wallis, D.D., F.R.S.*, (1616-1703). Pp. xi, 240. 12s. 6d. 1938. (Taylor and Francis)

A. W. Siddons and K. S. Snell. *Notes and answers to exercises in "A new geometry"*. Pp. 16. 1s. 1938. (Cambridge)

W. M. Smart. *Stellar dynamics*. Pp. viii, 434. 30s. 1938. (Cambridge)

P. J. Smith. *First year algebra*. Pp. 64, xv. 1s. 6d.; without answers, 1s. 3d.; answers separately, 8d. 1938. (McDougall)

J. M. Thomas. *Theory of equations*. Pp. x, 211. 12s. 1938. (McGraw-Hill)

W. G. Thomson. *A Scottish school geometry. IV*. Pp. viii, 431-516. 2s. N.d. (Oliver and Boyd, Edinburgh)

R. C. Tolman. *The principles of statistical mechanics*. Pp. xix, 661. 40s. 1938. International series of monographs on physics. (Oxford)

J. Trevelyan and J. Morley. *Functional arithmetic through citizenship. III. Ways and means*. Pp. vi, 90. 1s. 6d. 1939. (Longmans, Green)

S. Valentiner. *Vektoranalysis*. 5th edition. Pp. 136. RM. 1.62. 1938. Sammlung Götschen, 354. (Walter de Gruyter, Berlin)

G. Valiron. *Sur les valeurs exceptionnelles des fonctions méromorphes et de leurs dérivées*. Pp. 53. 18 fr. 1937. Actualités scientifiques et industrielles, 570; exposés sur la théorie des fonctions, IX. (Hermann, Paris)

C. de la Vallée Poussin. *Les nouvelles méthodes de la théorie du potentiel et le problème généralisé de Dirichlet*. Pp. 46. 15 fr. 1937. Actualités scientifiques et industrielles, 578; publications de l'Institut Mathématique de l'Université de Strasbourg, II. (Hermann, Paris)

F. Vasilescu. *La notion de capacité*. Pp. 49. 15 fr. 1937. Actualités scientifiques et industrielles, 571; exposés sur la théorie des fonctions, X. (Hermann, Paris)

A. Walker and G. P. McNicol. *A school algebra*. Pp. viii, 270, 60. 3s. 1938. (Longmans, Green)

C. Warrell. *Sane arithmetic for seniors. II*. Pp. 64. Limp cloth, 1s. 3d.; manilla, 1s. 1938. (Harrap)

A. N. Whitehead. *Modes of thought*. Pp. viii, 241. 7s. 6d. 1938. (Cambridge)

A. Wolf. *A history of science, technology and philosophy in the eighteenth century*. Pp. 814. 25s. 1938. (Allen and Unwin)

*American Mathematical Society Semicentennial publications. I. A semicentennial history of the American Mathematical Society, 1888-1938, with biographies and bibliographies of the past Presidents*. By R. C. Archibald. Pp. xi, 262. *II. Semicentennial addresses of the American Mathematical Society*. By E. T. Bell, J. F. Ritt, N. Wiener, E. J. McShane, T. Y. Thomas, R. L. Wilder, G. C. Evans, J. L. Synge, G. D. Birkhoff. Pp. 315. 1938. (American Mathematical Society, New York)

*Modern developments in fluid dynamics. I. II*. Composed by the Fluid Motion Panel of the Aeronautical Research Committee and others, and edited by S. Goldstein. Pp. xxiv, 330; xii, 331-702. 50s. 1938. Oxford Engineering Science series. (Oxford)

*The nineteen-thirty-eight mental measurements yearbook*. Edited by O. K. Buros. Pp. xiv, 415. \$3.1 1938. (Rutgers University Press, New Brunswick)

## EUREKA.

THE Archimedean, the undergraduate society of the University of Cambridge, is now a Junior Branch of the Mathematical Association and it is therefore with special pleasure that we welcome the first number of *Eureka*, the society's journal. Its 28 pages (6d.) are full of interest, both for what is printed and for what can be inferred concerning the ideas of the Cambridge mathematical undergraduates of to-day, from whose ranks will be drawn so many of the teachers of to-morrow. The spirit of the number is evidently that which holds "every man a debtor to his profession".

Two of our ex-Presidents contribute: Mr. Siddons a sketch of the history of our Association, Mr. Hope-Jones an anthem for the Archimedean.

The *Gazette* offers *Eureka* its congratulations on a good start and wishes it a successful career. A second number is to be published in May; copies may be had on application to D. J. Halliday, Clare College, Cambridge. The price is 6d., postage 1½d.

## INSTITUTE OF EDUCATION BULLETIN.

To think of the "Institute" is to think of Sir Percy Nunn. The first number of this *Bulletin* (1s.; Oxford University Press) contains as its main feature an article by Sir Percy on "The Institute and the training of teachers". No more need be said to commend this number to all those interested in the training of teachers of mathematics.

## FOR SALE.

Whitehead and Russell: *Principia Mathematica*. 2nd edition, 3 volumes. £4.

Goursat: *Cours d'analyse mathématique*. Cinquième édition. 3 volumes, strongly bound in black cloth with gilt lettering. £3.

Both works are in new condition.

Apply to J. B. Bretherton, 508 Manchester Road, Warrington, Lancs.

## LONDON BRANCH.

At a meeting at Bedford College, on 17th February, a discussion took place on "The Project method in the teaching of mathematics".

Mr. Max Black gave a general account of the aims of the method. He defined a project as a wholehearted, purposeful activity, proceeding in a social environment.

Miss Bowman illustrated the work done in infant and junior schools and showed that the method may be used to cover all the mathematical work usually taught. The speaker has arranged an exhibition of work done in this connection by the scholars.

Mrs. Williams dealt with the use of the Project in senior schools and gave details of actual projects. She pointed out some of the dangers inherent in the method.

An interesting discussion followed, which brought out other advantages and disadvantages of the method.

A. J. TAYLOR (*Hon. Sec.*).

## PLYMOUTH AND DISTRICT BRANCH.

At a meeting on the 21st March, 1939, a gathering of over twenty teachers and students unanimously adopted the resolution that—"Subject to the approval of the Council, a Plymouth and District Branch of the Mathematical Association be formed".

It was announced that the number of members likely to become attached to the Branch was of the order of fifteen, while some twenty-five associate members were expected to join.

A representative Committee of six was elected to assist the following officers:

*President*—F. Sandon.

*Vice-President*—G. H. Bonser.

*Hon. Secretary and Treasurer*—F. W. Kellaway.

The "business" portion of the meeting was followed by an animated discussion, opened by Mr. Sandon, on "The Spens Report in its application to Mathematics", the report receiving much constructive criticism.

A questionnaire has been issued regarding future programmes.

F. W. KELLAWAY, (*Hon. Sec.*).

## SHEFFIELD AND DISTRICT BRANCH.

A MEETING to inaugurate this Branch was held in the University on Tuesday, 29th November, 1938; there was an attendance of over forty. Officers were elected, the President being Professor Daniell, who then gave a talk on "Some doubts about mathematical teaching in the University".

The second meeting was held on 7th February, 1939. Mr. V. A. Carpenter read a paper on "Modern developments in marine propulsion", and Mr. A. Blackwell one on "Some doubts about mathematical teaching in the schools", in which he presented a strong case for drastic reform of present syllabuses.

The third meeting was held on 16th March, 1939, and was devoted to a vigorous discussion of the issues raised by Mr. Blackwell at the preceding meeting.

The present membership of the Branch is 29 members and 32 associates.

JOHN W. COWLEY (*Hon. Sec.*)

## SYDNEY BRANCH.

## REPORT FOR 1938.

THERE are at present 19 full members and 101 associate members. This represents a slight falling-off from last year. Circulars have been sent to a number of schools, inviting teachers to join the Association. It is expected that by this means the former membership will be again attained.

During the year the following meetings were held:

1. A meeting of the Organising Committee, which outlined the work for the year. Also at this meeting the mathematical papers set for the previous Intermediate and Leaving Certificate examinations were discussed.

2. A meeting held in the Electrical Engineering School, at which Mr. D. M. Myers gave a very interesting address on "The solution of differential equations by mechanical means". Two machines, designed by Mr. Myers, were used to demonstrate how solutions were obtained.

3. A meeting at which Mr. W. B. Smith-White gave an address on "The internal constitution of the stars".

## 4. The Annual Meeting, at which the main business was :

(i) Election of officers for 1939 : *President* : Mr. H. H. Thorne ; *Vice-President* : Professor E. M. Wellish ; *Joint Hon. Secretaries* : Miss E. A. West, Mr. H. J. Meldrum ; *Hon. Treasurer* : Mr. A. G. Aitkin.

(ii) An address by Mr. Thorne on "Approximations". Mr. Thorne showed how approximations could be obtained for  $\sqrt{N}$ ,  $\sqrt[3]{N}$ , the solution of algebraic equations of any degree,  $\sin x$ ,  $\cos x$ ,  $x!$ .

Professor Wellish spoke briefly about the proposals for the courses of study to be put into operation by the newly-formed Board of Secondary School Studies.

## BOOKS RECEIVED FOR REVIEW.

**P. B. Ballard and J. Brown.** *The London arithmetics.* Second series. Pupil's book IV. Pp. 80. Paper, 1s. 2d. ; limp cloth, 1s. 4d. N.d. (University of London Press)

**S. Beatty and J. T. Jenkins.** *Introduction to the calculus. I.* Pp. 650. 25s. 1938. (University of Toronto Press ; Humphrey Milford, Oxford)

**W. Blaschke.** *Ebene Kinematik.* Pp. 56. Geb. RM. 3.75 ; geh. RM. 3. 1938. *Hamburger mathematische Einzelschriften*, 25. (Teubner)

**E. Böhmer.** *Differenzengleichungen und bestimmte Integrale.* Pp. viii, 148. RM. 7.50. 1939. (Koehler, Leipzig)

**L. de Broglie.** *La Mécanique ondulatoire des systèmes de corpuscules.* Pp. vi, 223. 100 fr. 1939. *Collection de Physique mathématique*, 5. (Gauthier-Villars)

**E. Cartan.** *Leçons sur la théorie des Spineurs. II. Les spineurs de l'espace à  $n > 3$  dimensions. Les spineurs en géométrie riemannienne.* Pp. 96. 25 fr. 1938. *Actualités scientifiques et industrielles*, 701 ; *exposés de géométrie*, XI. (Hermann, Paris)

**A. Church.** *A bibliography of symbolic logic.* Reprinted from the *Journal of Symbolic Logic*. Two parts. Pp. 121-218, 178-212. \$2.50 ; all rag paper, \$3.25. In parts : \$1.50 ; all rag paper, \$2 : \$1 ; all rag paper, \$1.25. 1936, 1938 ; reprint 1939. (Association for Symbolic Logic, Brown University, Rhode Island)

**B. Cohn.** *Tables of addition and subtraction logarithms with six decimals.* 2nd edition, prepared by L. J. Comrie. Pp. viii, 63. 10s. 1939. (Scientific Computing Service)

**L. Crosland.** *Introductory school mathematics.* With answers. Pp. x, 208, xxxi. 2s. 6d. 1939. (Macmillan)

**F. Fitting.** *Panmagische Quadrate und magische Sternvielecke.* Pp. 70. RM. 3. 1939. *Scientia Delectans*, 4. (Koehler, Leipzig)

**B. J. Fulford.** *Higher Certificate and Intermediate tests in mathematics.* Pp. 90. 1s. 6d. 1938. (University Tutorial Press)

**J. S. Georges and J. M. Kinney.** *Introductory mathematical analysis.* Pp. xv, 605. 12s. 6d. 1938. (Macmillan)

**A. Gloden.** *Sur les égalités multigrades.* Pp. 91. 25 fr. belges. 1938. (Beffort, Luxembourg)

**H. K. Hughes and G. T. Miller.** *Trigonometry.* With tables. Pp. viii, 189, 79. 7s. 6d. 1938. (John Wiley, New York ; Chapman and Hall)

**G. Kowalewski.** *Die klassischen Probleme der Analysis des Unendlichen.* 3rd edition. Pp. viii, 404. RM. 10. 1938. (Koehler, Leipzig)

**L. J. Rouse.** *College algebra.* 2nd edition. Pp. xiii, 462. 11s. 1939. (John Wiley, New York ; Chapman and Hall)

H. Seifert und W. Threlfall. *Variationsrechnung im Grossen (Morsesche Theorie.)* Pp. 115. Geb. RM. 6.75; geh. RM. 6. 1938. Hamburger mathematische Einzelschriften, 24. (Teubner)

H. G. Smith. *Figuring with graphs and scales.* Pp. x, 62. 4s. 6d. 1938. (Stanford University Press; Humphrey Milford, Oxford)

G. H. Thomson. *The factorial analysis of human ability.* Pp. xv, 326. 16s. 1939. (University of London Press)

J. Tinbergen. *A method and its application to investment activity.* Pp. 164. 3s. 6d. 1939. Statistical testing of business-cycle theories, 1. (League of Nations, Geneva; Allen and Unwin)

C. Warrell. *Sane arithmetic for seniors. III.* Pp. 64. Manilla, 1s.; limp cloth, 1s. 3d. 1939. (Harrap)

L. Weisner. *Introduction to the theory of equations.* Pp. ix, 188. 10s. 1938. (Macmillan)

H. Weyl. *The classical groups. Their invariants and representations.* Pp. xii, 302. 18s. 1939. (Princeton University Press, New Jersey; Humphrey Milford, Oxford)

J. White. *ABC of geometry teaching. An exposition of Dr. Maria Montessori's Geometrical Pedagogy.* Pp. 87. 1s. 2d. N.d. Auto-education guides, 6. (Auto-education Institute)

*Research and statistical methodology books and reviews, 1933-1938.* Edited by O. K. Buros. Pp. vi, 100. \$1.25. 1938. (Rutgers University Press, New Brunswick)

July, 1939

## QUEENSLAND BRANCH.

### REPORT FOR THE YEAR 1938-1939.

SINCE last report we have to record the death of a member, Mr. J. G. Leadbeater, B.A., LL.B.

The 1938 Annual Meeting was held at the University on 8th April. The Annual Report and the Financial Statement for the year then ending were presented and were adopted and the Officers for the coming year were elected. Professor Simonds' Presidential Address was on the subject, "From Kepler to Newton".

During the year three general meetings have been held at the University. On June 10th Mr. F. W. James, B.Sc., read a paper on "Aerial Surveying". The meeting of 5th August was devoted to a discussion on "The Teaching of Junior and Senior Algebra", and on 28th October Mr. J. P. McCarthy, M.A., read a paper on "The use of Orthogonal Projections in School Mathematics". There was a fair attendance at all meetings.

The number of members of the Branch at present is 24, of whom 10 are members of The Mathematical Association. The Financial Statement shows a credit balance of £5 13s. 6d.

The copies of *The Mathematical Gazette* come to hand regularly and are circulated as usual.

The Officers of the Branch are: *President*: Professor E. F. Simonds; *Vice-Presidents*: Mr. S. Stephenson, Mr. I. Waddle; *Hon. Secretary and Treasurer*: Mr. J. P. McCarthy; *Members*: Miss E. H. Raybould, Miss W. Hossack, Mr. E. W. Jones, Mr. R. A. Kerr, Mr. J. C. Deeney.

J. P. MCCARTHY (Hon. Sec.).

### EXCHANGE WANTED.

THE secretaries have particulars from a member in Johannesburg who wishes to arrange an exchange from January to December, 1940, with a teacher of mathematics in England. Any member who is interested is asked to communicate with Mrs. E. M. Williams (155 Holden Road, Woodside Park, London, N. 12).

### FOR SALE.

ELECTRIC calculating machine. Monroe  $10 \times 10 \times 20$ . Semi-automatic, and extra register. First-class condition. £30. L. M. Milne-Thomson, Gothic House, Maze Hill, London, S.E. 10.

### BOOKS RECEIVED FOR REVIEW.

R. C. Archibald. *Outline of the history of mathematics*. 4th edition. Pp. 66. 50 cents. 1939. (Mathematical Association of America, Oberlin, Ohio)

L. Bachelier. *Les nouvelles méthodes du calcul des probabilités*. Pp. viii, 69. 25 fr. 1939. (Gauthier-Villars)

C. B. Boyer. *The concepts of the calculus. A critical and historical discussion of the derivative and the integral*. Pp. vi, 346. 18s. 6d. 1939. (Columbia University Press, New York; Humphrey Milford)

D. Brunt. *Physical and dynamical meteorology*. 2nd edition. Pp. xxiv, 428. 25s. 1939. (Cambridge)

R. S. Burington and C. C. Torrance. *Higher mathematics with applications to science and engineering*. Pp. xiii, 844. 30s. 1939. (McGraw-Hill)

C. H. Currier, E. E. Watson and J. S. Frame. *A course in general mathematics*. Pp. ix, 382. 13s. 1939. (Macmillan)

K. Dörge. *Wahrscheinlichkeitsrechnung für nichtmathematiker*. Pp. 113. RM. 6. 1939. (Walter de Gruyter, Berlin)

- C. V. Durell. *A new geometry for schools. Stage A and Stage B.* Pp. xvi, 571, xxii. 5s. 6d. Stage A: 1s. 6d. Stage B, Parts I-III: 4s. 6d. Stage B, Part I: 1s. 9d. Part II: 2s. 6d.; Part III: 1s. 6d.; Parts I and II: 3s. 9d. 1939. (Bell)
- C. M. Fleming. *Beacon arithmetic.* Book I, parts 1 and 2; Book II, parts 1 and 2; Book III, parts 1 and 2. Each part, 128 pp. Each part, 1s. 9d. 1939. (Ginn)
- C. M. Fleming. *Beacon arithmetic. Teachers' manual with diagnostic tests.* Pp. viii, 165. 3s. 6d. 1939. (Ginn)
- C. M. Fleming and E. H. Grassam. *Beacon number reader.* Pp. 96. 1s. 6d. 1939. (Ginn)
- S. L. Green. *The theory and use of the complex variable. An introduction.* Pp. viii, 136. 10s. 6d. 1939. (Pitman)
- C. G. Hayter and M. J. G. Hearley. *A first geometry.* Pp. 193. Without answers, 2s. 9d.; with answers, 3s. 1939. (Harrap)
- Sir Thomas Heath. *Greek mathematics and astronomy.* Reprinted from *Scripta Mathematica*. Pp. 215-232. 25 cents. (New York)
- D. Hilbert und P. Bernays. *Grundlagen der Mathematik. II.* Pp. xii, 498. Geb. RM. 43.80; geh. RM. 42. 1939. Die Grundlehren der mathematischen Wissenschaften, 50. (Springer, Berlin)
- C. J. Keyser. *A mathematical prodigy: history and legend.* Reprinted from *Scripta Mathematica*. Pp. 12. 20 cents. (New York)
- H. Levy. *Modern science. A study of physical science in the world today.* Pp. x, 736. 21s. 1939. (Hamish Hamilton)
- N. W. McLachlan. *Complex variable and operational calculus with technical applications.* Pp. xi, 355. 25s. 1939. (Cambridge)
- E. von Mises. *Probability, statistics and truth.* Translated by J. Neyman, D. Sholl and E. Rabinowitsch. Pp. xvi, 323. 12s. 6d. 1939. (Hodge)
- E. H. Moore. *General analysis. II. The fundamental notions of general analysis.* Pp. vi, 255. 14s. 1939. Memoirs of the American Philosophical Society, Vol. I, Part 2. (American Philosophical Society, Philadelphia; Humphrey Milford)
- M. H. A. Newman. *Elements of the topology of plane sets of points.* Pp. viii, 221. 12s. 6d. 1939. (Cambridge)
- F. Perrin. *Mécanique statistique quantique.* Pp. 224. 100 fr. 1939. *Traité du calcul des probabilités*, tome II, fasc. 5. (Gauthier-Villars)
- O. Perron. *Irrationalzahlen.* 2nd edition. Pp. viii, 199. RM. 9.80. 1939. Göschens Lehrbücherei, 1. (Walter de Gruyter, Berlin)
- P. B. Rider. *An introduction to modern statistical methods.* Pp. ix, 220. 13s. 6d. 1939. (John Wiley, New York; Chapman and Hall)
- A. Scholz. *Einführung in die Zahlentheorie.* Pp. 136. RM. 1.62. 1939. Sammlung Göschens, 1131. (Walter de Gruyter, Berlin)
- D. E. Smith. *Addenda to "Rara Arithmetica".* Pp. x, 52. 10s. 1939. (Ginn)
- J. Solomon. *Protons, Neutrons, Neutrinos.* Pp. xiii, 228. 100 fr. 1939. *Collection de Physique mathématique*, 6. (Gauthier-Villars)
- W. G. Thomson. *A Scottish school geometry. II. III.* Pp. 247-324, 325-428. 1s. 6d.; 1s. 9d. N.d. (Oliver and Boyd)
- E. C. Titchmarsh. *The theory of functions.* 2nd edition. Pp. x, 454. 25s. 1939. (Oxford)
- A. A. Tschuprow. *Principles of the mathematical theory of correlation.* Translated by M. Kantorowitsch. Pp. x, 194. 12s. 6d. 1939. (Hodge)
- W. Weber. *Die Pellsche Gleichung.* Pp. vii, 151. RM. 5. 1939. *Deutsche Mathematik*, Beiheft 1. (Hirzel, Leipzig)
- H. Wieleitner. *Geschichte der Mathematik. I. Von der ältesten Zeiten bis zur Wende des 17. Jahrhunderts.* New impression. Pp. 136. RM. 1.62 *II. Von 1700 bis zur Mitte des 19. Jahrhunderts.* New impression. Pp. 154. RM. 1.62. 1939. Sammlung Göschens, 226, 875. (Walter de Gruyter, Berlin)

H. Wieleitner. *Algebraische Kurven. II. Allgemeine Eigenschaften.* New impression. Pp. 122. RM. 1.62. 1939. Sammlung Götschen, 436. (Walter de Gruyter, Berlin)

R. Woods. *Analytic geometry.* Pp. xiii, 294. 10s. 1939. (Macmillan)

## BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, i.e. volume, page, and number. The names of those sending the questions will not be published.

The Secretary would be glad to receive any solutions that have not yet been returned.

## JOURNALS RECEIVED.

*When no number is attached, no part has been received since a previous acknowledgment.*

- Abhandlungen aus dem Math. Seminar der Hamburgischen Univ. → 12 : 3-4
- American Journal of Mathematics → 61 : 1, 2
- American Mathematical Monthly. [2 : 6, 11] → 46 : 5
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- Annales de la Société Polonaise de Mathématique → 17 : 1
- Annali della R. Scuola di Pisa. Ser. 2 → 8 : 2
- Annals of Mathematics → 40 : 2
- Berichte ... der Akad. der Wiss. zu Leipzig : Math.-Phys. Klasse
- Boletín Matemático [11 : 12] → 12 : 6
- Boletín del Seminario Matemático Argentina → 19, 23
- Bollettino della Unione Matematica Italiana. Ser. 2 → 1 : 2
- Bulletin of the American Mathematical Society → 45 : 4
- Bulletin de l'Association des Professeurs de Mathématiques. (98) → 108
- Bulletin of the Calcutta Mathematical Society → 30 : 3-4
- Bulletin de l'Académie ... Royale Serbe. A. → 4
- Communications ... de Kharkoff. Ser. 4. 15 : 1, 2
- Contribución al Estudio de las Ciencias Fís. y Mat. Serie Técnica. [1 : 5] → 5 : 2
- Duke Mathematical Journal → 5 : 1
- L'Enseignement Mathématique → 37 : 5-6
- Esercitazioni Matematiche. Ser. 2 → 11 : 5-8
- Gazeta Matematica → 44 : 10
- Half-Yearly Journal of the Mysore University
- Jahresbericht der Deutschen Math.-Verein. (1-4 ; 11-22 ; 30) → 48 : 9-12
- Japanese Journal of Mathematics → 15 : 3

- Journal of the Faculty of Sciences, Hokkaido → 7 : 3-4  
 Journal of the Indian Mathematical Society → N.S. 3 : 5  
 Journal of the London Mathematical Society → 14 : 1  
 Journal of the Mathematical Association of Japan → 21 : 3  
 Journal de l'Institut Mathématique d'Ukraine (1935) → 1938 : 1  
 Mathematica elemental 5 : 1, 2 ; 7 : 1-6  
 Mathematical Notes  
 Mathematics Student → 6 : 3  
 Mathematics Teacher [17 : 1, 3] → 32 : 5  
 Monatshefte für Mathematik und Physik. (33) → 47 : 2  
 National Mathematics Magazine (Louisiana). (11) → 13 : 7  
 Nieuw Archief voor Wiskunde → 19 : 3-4  
 Nieuwe Opgaven  
 Periodico di Matematiche. Ser. 4. [13 : 2—15; 18 : 2] → 19 : 2  
 Proceedings of the Edinburgh Mathematical Society → Ser. 2. 5 : 3  
 Proceedings of the London Mathematical Society → Ser. 2. 44 : 6  
 Proceedings of the Physico-Mathematical Society of Japan. Ser. 3 → 21 : 3  
 Publicaciones de la Facultad de Ciencias Físico-Matemáticas Universidad Nacional de la Plata. 115, 117, 118  
 Publications Mathématiques de Belgrade → 6-7  
 Publications de la Fac. des Sc. de Masaryk. 261, 262, 265, 266, 270  
 Recueil Mathématique (Moscow). N.S. → 4 : 2  
 Rendiconti del Circolo di Palermo. (26) [45-50] → 61 : 2  
 Rendiconti del Seminario di Cagliari → 9 : 2  
 Rendiconti del Seminario di Milano → 11  
 Report of the National Physical Laboratory. 1937, 1938  
 Revista de Ciencias (Peru). (371) → 426  
 Revista Matemática Hispano-Americana (Madrid). 11 : 1-2; 13 : 1-6  
 School Science and Mathematics. (9) [9 : 3; 17 : 3; 18 : 1; 19 : 5; 38 : 2, 8] → 39 : 5  
 Science Progress. (101) → 132  
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 Sitzungsberichte der Berliner Mathematischen Gesellschaft. 37  
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 Travaux de l'Institut Mathématique d'Ukraine. 1  
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## THE LIBRARY.

THE RED HOUSE, SONNING-ON-THAMES.

THERE are abundant reasons for maintaining the Library service, but a tightening of regulations is inevitable and their strict observance is essential. In the past the administration of the Library has been free and easy. If one member is making use of a book, why stickle for its return unless another member is waiting? Who benefits by the mere presence of a volume on a library shelf? If you know where a book is, why waste time and trouble on formal recalls and renewals? If it is true that sometimes the borrower has put the book aside and forgotten it until a belated reminder has extracted the book complete with apology, who has been a penny the worse off in the long run?

In an unsettled community, the conduct which prompts such questions as these is no longer excused by them. Books may be lost to the Library not from neglect or wanton misuse, but because in a domestic upheaval or a mobilisation no one can be expected to treat a book whose intrinsic value is perhaps comparatively small as a major responsibility or even to call it to mind at all. The chance that a book or a borrower is not at the address recorded in the Library must be reduced. The chance that a borrower is forgetful must be eliminated. The chance that a book left behind at a critical moment will nevertheless reach the Library again in due course must be increased.

(1) In future a loan will be for one month only in the first place, and must be formally renewed at monthly intervals if a book is to be kept. Only in this way can the Librarian be satisfied that he is not losing contact with the books in his charge, and unless members will cooperate by complying scrupulously with this requirement the Library service must be suspended. No librarian can undertake to discriminate by compiling a black list, a white list, and lists of various shades of grey; defaulters will close the Library, not to themselves only but altogether, for the duration of the war.

(2) Every borrower is urged to return each book as soon as he has done with it, irrespective of the date when the loan formally expires.

(3) Every volume issued during the war will be in a distinctive skeleton jacket bearing the name of the Association and the address of the Library as well as the name of the borrower. This should facilitate recognition and dispatch in an emergency, by the borrower or by anyone by whom his affairs have to be put in order. The fitting of these jackets is the Librarian's contribution to the effort to carry on, and his justification for insisting that members must play their part too.

(4) Books will not be sent out of the United Kingdom, even to Ireland.

(5) One form of activity must cease. The Librarian has often negotiated loans of books which the Association did not possess, pledging the credit of the Association for their safe return to the owners. Under present conditions it is impossible to commit the Association to this indirect responsibility.

THE Librarian reports gifts as follows :

From Mr. **W. Hope-Jones**, on the conclusion of his year of office as President :

**W. A. WHITWORTH**

Choice and Chance (5 (1901) rep.) - - - - 1934

From Miss **M. J. Griffith** :

**I. NEWTON**

Principia ( ) II-IV - - - - 1822

*The Glasgow reprint of the Jesuits' edition.*

*The donor wishes to complete the set and any member who comes across an odd copy of Vol. I for sale will do the Association a good turn by reporting it to the Librarian.*

From Mr. **W. Hope-Jones**, a number of school books, together with

LORD HALDANE	Reign of Relativity (2)	-	-	-	-	-	1921
M. HOPKINS	Chance and Error	-	-	-	-	-	1923
E. J. ROUTH	Dynamics of a Particle	-	-	-	-	-	1898

From Mr. **C. L. M. Jockel**:

E. ROUCHÉ et L. LÉVY	Analyse Infinitésimale (2 vols.)	-	-	-	-	1900, 1902
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From Mr. **C. A. B. Smith**:

J. H. JOHNSTON	Reverse Notation	-	-	-	-	-	[1938]
	Arithmetic with positive and negative digits.						

From Mr. **F. J. Swan**:

S. F. LACROIX	Traité du Calcul Différentiel et du Calcul Intégral	(2 vols.)	1797, 1798
	Traité des Différences et des Séries	-	1800
	Described in this edition as a sequel, but in the second edition as the third volume of the <i>Traité du Calcul</i> .		

The following have been bought :

JAMES BERNOULLI	Über unendliche Reihen (1689-1704)	Ostwald 171	1909
	Translated from Latin and edited by G. Kowalewski.		
H. G. ZEUTHEN	Die Lehre von den Kegelschnitten im Altertum	-	1886
	Translated from Danish into German by R. v. Fischer-Benzon.		

#### SOUTHAMPTON BRANCH.

DURING the past academic year, under the Presidency of Miss A. M. Trout, B.A., the Branch has been very active. Mr. A. Prag, of the New Herrlingen School, spoke to us in October on "Some Striking Features of 17th-Century Mathematics". It was a pleasure to hear a distinguished mathematician and historian speak on this subject. In November Mr. C. O. Tuckey, M.A., spoke to us on "The Geometry Report", and in December our President fascinated us with an expert account of "Mediaeval Reckonings of Time".

In February Mr. L. B. Benny, M.A., Principal of Portsmouth Municipal College, spoke on "The Almagest of Ptolemy, and the Beginnings of Trigonometry". We are looking forward to hearing Mr. Benny again in November. Mr. W. O. Storer, B.A., was very entertaining on "Thinking made Foolproof" in March, and Mr. C. O. Tuckey aroused a great deal of discussion in May by his presentation of "A Universal Crib to Trigonometry".

We visited the R.A.F. Meteorological Station at Calshot in June, in chartered motor-launches, and spent an interesting and happy afternoon there.

Our membership has increased during the year, and we feel that we can act as a very useful centre for those teachers in Hampshire, Wiltshire and Dorset interested in mathematics and the teaching of mathematics. Our best attendances at open lectures have been over 70. Lest members of the Association do not know this, it costs them only a shilling to join our Branch!

Mr. A. Robson, M.A., of Marlborough College, has kindly consented to head our programme on 17th October with a talk on "The First Year of Coordinate Geometry in the Sixth Form".

D. PEDOE (*Hon. Sec.*), University College, Southampton.

## BRANCH REPORTS

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### VICTORIA BRANCH.

#### REPORT FOR 1938.

THE number of full members has increased from 6 to 13, and that of Associates from 16 to 26. Our Assets amount to £15 13s. 9d., including stocks of stationery, but not including books in the Library, most of which are the gift of the University Maths. Department, through Professor Cherry. We regret the death of Miss Helen Barnard, a former member of the Branch, and daughter of the original Secretary and former President. Miss Flynn, after many years of service as Vice-President, has decided to relinquish office. The Branch offers her its hearty thanks for her continued activities and its best wishes for the future. We are pleased to welcome back Miss Kellaway after her trip abroad, and hope to hear from her rather fully during the year.

During this year there were five meetings and one broadcast, as follows :

March : Mr. H. B. Sarjeant spoke on " Mathematics and Gambling ", and the interest in this was not confined to the animated discussion at the meeting but, stimulated by metropolitan and interstate Press reports, continued, by correspondence, for some months.

May : Professor A. Burstall addressed a joint meeting of the Association and the University Mathematical Society on " Some simple Applications of Mathematics to Engineering Design ", after which the Association entertained the Society at supper.

June : Professor Cherry caused our eyes to see all sorts of colours when he set some black-and-white designs rotating. Then Mr. Palfreyman showed how to simplify logarithmic calculations by writing  $10^x$  as  $[x]$ , exemplifying it in a self-checking method for solving irreducible cubic equations.

Typographed notes of this were sent, on application, throughout the State, and also to Sydney.

July : Mr. R. S. Clayton, a real, live stockbroker gave us the " good oil " on Stocks and Shares.

August : Messrs. A. McDonell and F. R. Manley broadcast from 3LO on " Decimal Coinage ".

October : Professor Michell communicated a paper, " The solution of cubic equations by means of Epicene Functions ".

Mr. Belz discussed the question of the equality of  $(Z^w)^z$  and  $Z^{wz}$ .

Professor Cherry investigated the number of solutions of  $f(x, y) = 0 = F(x, y)$  where  $f$  and  $F$  are non-homogeneous, rational, integral, algebraic functions of degree  $m$  and  $n$ .

These papers were answers to Problems handled by the local Bureau.

Full notes of the broadcast and the papers read in October are preserved and may be inspected by anyone interested.

The problem Bureau and the Junior Association merely continue. They do not flourish.

F. J. D. SYER (*Hon. Sec.*)

### YORKSHIRE BRANCH.

At the Summer Meeting of the Yorkshire Branch, held at Rothwell Grammar School, Mr. E. R. Clarke, M.A., B.Sc., LL.B., M.Ed., Headmaster of Aireborough Grammar School, opened a discussion on the Spens Report on Secondary Education. Five resolutions were put forward and, after a long discussion, the following resolutions were passed :

1. No attempt is made in the Report to sort out different teaching methods as applicable to different ranges of ability and the main suggestions of the

Report seem directed to the teaching of pupils rather below the general average of Grammar Schools.

2. That the realistic treatment of mathematics of Technical Schools is unnecessary and unsuited to the more gifted pupils of Grammar Schools.

3. We do not think that there is any indication in the Spens Report or elsewhere that the time allotted to mathematics can be reduced without loss of efficiency.

4. The idea of teaching mathematics in its historical order is inadvisable.

H. BEARDWOOD (*Hon. Sec.*)

## BOOKS RECEIVED FOR REVIEW.

H. V. Allen. *Commercial arithmetic*. Pp. ix, 455. 6s. 1939. (Longmans, Green)

W. W. Rouse Ball. *Mathematical recreations and essays*. 11th edition. Revised by H. S. M. Coxeter. Pp. xvi, 418. 10s. 6d. 1939. (Macmillan)

F. Bon. *Ist es Wahr dass  $2 \times 2 = 4$  ist? II. Von den Kriterien der Wahrheit*. Pp. 83. III. *Von den mathematischen Grundbegriffen*. Pp. 80. N.p. 1939. (Leipzig)

W. G. Borchardt. *A School Certificate Trigonometry*. Pp. vii, 307, xxxiii, xxxix. 4s. 1939. (Rivingtons)

C. V. Durell. *Exercises and theorems in geometry*. Pp. xvi, 568, xxii. 5s. 6d.; Stage A: 1s. 6d.; Stage B, I-III: 4s. 6d.; Stage B, I: 1s. 9d.; II: 2s. 6d.; III: 1s. 6d.; I-II: 3s. 9d. 1939. (Bell)

H. Freeman. *Mathematics for actuarial students*. I. *Elementary differential and integral calculus*. Pp. viii, 183. 9s. II. *Finite differences, probability and elementary statistics*. Pp. xiii, 339. 25s. 1939. (Institute of Actuaries; Cambridge University Press)

C. H. Goulden. *Methods of statistical analysis*. Pp. vii, 277. 17s. 6d. 1939. (John Wiley, New York; Chapman and Hall)

J. Hlavaty. *Differentialgeometrie der Kurven und Flächen und Tensorrechnung*. Übersetzung von M. Pinl. Pp. xi, 569. Fl. 14; geb. fl. 15.50. 1939. (Noordhoff, Groningen)

C. E. Van Horn. *A preface to mathematics*. Pp. xii, 124. \$2.50. 1938. (Chapman and Grimes, Boston, Mass.)

R. T. Hughes. *Arithmetic*. Pp. xv, 436. Without answers, 5s.; with answers, 5s. 6d. In parts: I, II, III without answers, each 2s. 3d.; with answers, each 2s. 6d.; I and II without answers, 4s.; with answers, 4s. 6d.; II and III without answers, 4s.; with answers, 4s. 6d. 1939. The London Mathematical series. (University of London Press)

D. Humphrey and E. A. Baggott. *Elementary mechanics with hydrostatics*. Pp. xvi, 628. 8s. 1939. (Longmans, Green)

C. J. Keyser. *The meaning of mathematics*. Offprint. Pp. 15-28. 20 cents. N.d. (*Scripta Mathematica*, New York)

V. A. Kostitzin. *Mathematical biology*. Translated by T. H. Savory. Pp. 238. 7s. 6d. 1939. (Harrap)

L. Lines. *Intermediate solid geometry*. Chapters I-VIII of *Solid geometry*. Pp. xv, 131. 2s. 6d. 1939. (Macmillan)

R. C. Lyness and E. R. Emmet. *An introduction to economics*. Pp. viii, 372. 4s. 6d. 1939. (Bell)

F. H. Miller. *Calculus*. Pp. xiv, 419. 15s. 1939. (John Wiley, New York; Chapman and Hall)

L. F. Richardson. *Generalized foreign politics*. Pp. 91. 8s. 6d. 1939. *British Journal of Psychology*, Monograph supplement, 23 (Cambridge)

# INDEX TO VOLUME XXIII.

THE index to Volume XXIII will be circulated with the *Gazette* for February, 1940.

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- Jahresbericht der Deutschen Math.-Verein. [10, 23-29] → 49 : 1
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- Journal of the Indian Mathematical Society → N.S. 3 : 6
- Journal of the London Mathematical Society → 14 : 4
- Journal of the Mathematical Association of Japan → 21 : 4
- Journal de l'Institut Mathématique d'Ukraine
- Matematica elemental 6 : 1-6
- Mathematical Notes
- Mathematics Student → 7 : 1
- Mathematics Teacher [17 : 1, 3] → 32 : 6
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- National Mathematics Magazine (Louisiana). (11) → 14 : 1
- Nieuw Archief voor Wiskunde

## Nieuwe Opgaven

Periodico di Matematiche. Ser. 4. [13:2—15; 18:2] → 19:4

Proceedings of the Edinburgh Mathematical Society → Ser. 2. 6:1

Proceedings of the London Mathematical Society → Ser. 2. 45:6

Proceedings of the Physico-Mathematical Society of Japan. Ser. 3 → 21:8

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Rendiconti del Seminario di Cagliari → 9:3

Rendiconti del Seminario di Milano

Report of the National Physical Laboratory

Revista de Ciencias (Peru). (371) → 428

Revista Matemática Hispano-Americana (Madrid)

School Science and Mathematics. (9) → 39:6

Science Progress. (101) → 134

Scripta Mathematica → 6:1

Sitzungsberichte der Berliner Mathematischen Gesellschaft

Studia Mathematica

Travaux de l'Institut Mathématique de Tbilissi

Travaux de l'Institut Mathématique d'Ukraine

Universidad (Zaragoza). (9) → 16:3

Unterrichtsblätter für Mathematik und Naturwissenschaften

Wiskundige Opgaven met de Oplossingen. 17:2

## BOOKS RECEIVED FOR REVIEW.

A. C. Aitken. *Determinants and matrices*. Pp. vii, 135. 4s. 6d. 1939. University mathematical texts, 1. (Oliver and Boyd)

A. A. Albert. *Structure of algebras*. Pp. xi, 210. \$4. 1939. American Mathematical Society Colloquium Publications, 24. (American Mathematical Society)

H. S. Allen and R. S. Maxwell. *A text-book of heat*. II. Pp. ix, 531-848, xi. 10s. 6d. 1939. (Macmillan)

B. B. Baker and E. T. Copson. *The mathematical theory of Huygens' Principle*. Pp. vii, 155. 12s. 6d. 1939. (Oxford)

R. H. Baker. *Astronomy*. 3rd edition. Pp. x, 527. 16s. 1939. (Macmillan)

S. Barnard and J. M. Child. *Advanced algebra*. Pp. x, 280. 16s. 1939. (Macmillan)

C. Bragdon. *A primer of higher space. The fourth dimension*. Pp. 81. 5s. 1939. (Dakers)

G. W. Caunt. *Elementary calculus*. Pp. 388. 7s. 6d. 1939. (Oxford)

S. Chapman and T. G. Cowling. *The mathematical theory of non-uniform gases*. Pp. xxiii, 404. 30s. 1939. (Cambridge)

- Sir Arthur Eddington.** *The philosophy of physical science.* Pp. ix, 230. 8s. 6d. 1939. (Cambridge)
- A. Geary, H. V. Lowry and H. A. Hayden.** *Mathematics for technical students. II.* Pp. viii, 417. With tables. 5s. 1939. (Longmans)
- B. P. Gillespie.** *Integration.* Pp. viii, 126. 4s. 6d. 1939. University mathematical texts, 3. (Oliver and Boyd)
- K. R. Gunjekar.** *An introduction to the calculus.* Pp. xiv, 341. Rs. 4 (Indian). 1938. (Oxford)
- J. H. Gunlake.** *Premiums for life assurance and annuities.* Pp. xi, 126. **C. F. Wood.** *The treatment of extra risks.* Pp. 71. In one volume. 9s. 6d. 1939. (Institute of Actuaries Students' Society; Cambridge University Press)
- P. J. Haler and A. H. Stuart.** *First course in mathematics for technical students.* 3rd edition. Pp. vii, 295. 3s. 1939. (University Tutorial Press)
- E. L. Ince.** *Integration of ordinary differential equations.* Pp. viii, 148. 4s. 6d. 1939. University mathematical texts, 4. (Oliver and Boyd)
- V. S. Mallory.** *The relative difficulty of certain topics in mathematics for slow-moving ninth grade pupils.* Pp. x, 179. \$2.10. 1939. (Teachers College, Columbia University, New York)
- J. C. P. Miller.** *Tables for converting rectangular to polar co-ordinates.* Pp. 16. 2s. 1939. (Scientific Computing Service, 23 Bedford Square, London, W.C. 1)
- P. K. Rees and F. W. Sparks.** *College algebra.* Pp. xi, 312. 12s. 6d. 1939. (McGraw-Hill)
- D. E. Rutherford.** *Vector methods applied to differential geometry, mechanics, and potential theory.* Pp. viii, 127. 4s. 6d. 1939. University mathematical texts, 5. (Oliver and Boyd)
- E. Sevin.** *Physique stellaire. Essai de synthèse.* Pp. 80. 25 fr. 1939. (Gauthier-Villars)
- L. S. Shively.** *An introduction to modern geometry.* Pp. xi, 167. 10s. 1939. (John Wiley, New York; Chapman and Hall)
- A. W. Siddons and K. S. Snell.** *Introduction to geometry.* Pp. vi, 166. 2s. 3d. 1939. (Cambridge)
- A. W. Siddons and K. S. Snell.** *Notes and answers to exercises in "Introduction to geometry".* Pp. 24. 1s. 1939. (Cambridge)
- L. G. Simons.** *Fibre and mathematics, and other essays.* Pp. v, 101. \$1. 1939. The Scripta Mathematica Library, 4. (Scripta Mathematica, New York)
- D. E. Smith.** *Thomas Jefferson and mathematics.* 2nd edition. Offprint. Pp. 16. 25 cents. N.d. (Scripta Mathematica, New York)
- H. Steinhaus.** *Mathematical snapshots.* Pp. 135. 10s. 6d. 1938. (Stechert)
- G. Szegő.** *Orthogonal polynomials.* Pp. ix, 401. \$6. 1939. American Mathematical Society Colloquium Publications, 23. (American Mathematical Society)
- G. P. Thomson and W. Cochrane.** *Theory and practice of electron diffraction.* Pp. xi, 334. 18s. 1939. (Macmillan)
- J. Ville.** *Étude critique de la notion de Collectif.* Pp. 144. 75 fr. 1939. Monographies des probabilités, fasc. III. (Gauthier-Villars)
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